Inference with Attribute Grammars

Afgeleid door Attributengrammatica's

(met een samenvatting in het Nederlands)

PROEFSCHRIFT

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Preface

Abstraction is a programmer's best friend. From *TEX*, one of Knuth's inventions, I learned that abstraction is important. I am glad that I do not need to know implementation details of most of the type setting packages that I used for this thesis. TEX also taught me that it is sometimes necessary to get your hands dirty in order to obtain new abstractions. From *attribute grammars*, another invention of Knuth, I learned how beneficial abstraction facilities of programming languages can be to the development of software.

Attribute Grammars (AGs) offer attributes as a powerful abstraction mechanism for the description of computations that traverse trees. Since many programming tasks can be expressed as a computation over a tree, I often reason in terms of trees with attributes and relations between these attributes. This thesis offers an exploration of this mental model, and gives the reader the opportunity to embrace this model too.

Programs may conceptually be formulated as an attribute grammar, yet it may be hard to formalize such programs as an AG. During my master studies, I implemented an experimental type inferencer for uniqueness types in the Utrecht Haskell Compiler [Dijkstra et al., 2009] with attribute grammars using UUAG [Löh et al., 1998]. However, several aspects of the inference algorithm are not straightforward to describe with an attribute grammar because these aspects required fixpoint iteration and dynamic tree construction in their implementation. This is unfortunate, because UUAG provides many notational conveniences that would have saved me from writing tons of boilerplate code.

I worked on many language extensions to attribute grammars that make it easier to describe type inference algorithms by an attribute grammar. During my thesis project, it occurred to me that in order to express such algorithms, details of the actual evaluation of the grammar need to be exposed. But how to expose such details, without destroying the attractive abstraction mechanisms offered by the AG programming model? This thesis presents various solutions to this challenge.

Thesis process. The standard way to describe type systems is by giving a collection of inference rules. In the initial years of the project, I had lengthy discussions with my supervisors about patterns in these descriptions, and how to abstract from such patterns. During these discussions, it became clear that we needed an execution model for the symbols that we were scribbling on whiteboards.

It is not at all obvious how to formalize inference algorithms. Algorithms can typically be expressed nicely in a functional programming language. However, there are many implementation techniques that offer even higher levels of abstraction. Monads, constraint handling rules, and attribute grammars are implementation techniques with attractive properties. Combining these techniques, however, is not at all trivial.

In the later years of the project, I therefore narrowed my focus on exactly this challenge.

The original purpose of the Ruler project is to annotate type rules with sufficient information to derive the type inference algorithm mechanically. This requires a way to identify chunks of AG evaluation, which is what visits represent in ordered attribute grammars. The language RulerCore and this thesis emerged from being able to programmatically manipulate these visits.

Intended audience. This thesis is written with two types of audience in mind: an audience with a background in type systems, and an audience with a background in grammars. All in all, this thesis needs and explains more of attribute grammars than of type systems.

Acknowledgements. I would like to thank my promotor Doaitse Swierstra. Despite his busy life, Doaitse takes the time to spread his brilliant ideas and interesting anecdotes. I see Doaitse as a great advocate of functional programming, although our discussions usually started with an account of the low-level details of the tools that he fiddled with the day before. Nevertheless, Doaitse sets a great example, and it was an honor to work under his supervision.

My copromotor Atze Dijkstra is another inspiring example. It is amazing how Atze approaches problems with clear and concrete solutions, yet reasons at a high level of abstraction. Atze took a customer role in my research project, which motivated me a lot to make changes to the UUAG system. Atze has a calm appearance, yet I also experienced some of his adventurous and active traits. In particular, I recall early-morning training runs during conferences, at times that others are still deep asleep.

Valuable feedback was given by the reading committee (fill in later), and by the anonymous reviewers of the papers from which this thesis is composed.

The Software Technology group at Utrecht is a pleasant group to be part of, and I thank all my colleagues and former colleagues for the enjoyable time. I especially recall the crazy group effort for the ICFP contest in 2007, and the many Thursday-evening drinks.

During many discussions related to the UHC project, Jeroen Fokker took heroic efforts to keep our discussions concrete and focussed. Jeroen was very helpful in many organizational matters, and in fact, I learned programming with a copy of his lecture notes when I was still in secondary school.

I thank my former room mates Eelco Dolstra, José Pedro Magalhães, and Stefan Holdermans. I hope that Stefan does not tear his eyes out if he spots errors in the Dutch summary at the back of this thesis. I also had pleasant discussions with my room mates Jeroen Bransen and Alexander Elyasov. I hope that you will enjoy attribute grammars as much as I do.

Halfway in my thesis project, I worked for half a year in Brazil. I want to heartily thank Lucília Camarão de Figueiredo for her effort in making my stay in Brazil pleasant. It is a pity that we did not have more opportunities to work together. My greetings to your family and friends, because of the time they spend with me, and also to Elton and the others that were with me in the lab.

The last year, I worked together with Wishnu Prasetya and Jurriaan Hage to statically instrument ActionScript bytecode. It was certainly not easy to combine writing a thesis with developing a framework for bytecode manipulation. However, the use of Haskell and attribute grammars turns out to be a good choice so far. I know Jurriaan already since when I still was

a second year student. He creates a warm atmosphere in the group. When it is time for a break, Jurriaan is around for a chat about all sorts of subjects.

The Universität Freiburg, and the ProgLang group in particular, generously providing me with a place to work each time I visited Freiburg. I actually performed large chunks of the programming and paper writing for this thesis during my many short visits. *Herzlichen Dank!* Peter Thiemann invited me several times for some sports after work. Peter is very disciplined, which meant that the training sessions were nice, but hard.

With the *Broken Dagger* group, we had very nice geeky dinners after work, and necessary distraction in the form of the Ars Magica games that we are still playing remotely. It is surprising what hidden personalities colleagues expose when exploring a fantasy setting. Thank you Andres Löh for taking this initiative. By the way, the characters will next session stumble by accident on a magic book about attribute grammars that drives them insane...

The rather secretive IRC channel `#klaplopers`, formed by former occupants of the ST-Lab, provided welcoming distractions, both online and in real life. The channel is becoming somewhat silent as more participants have a reduced online presence. Unfortunately, I was also too occupied with my thesis to participate actively. I want to thank Arthur van Dam and others for several nice mountainbike trips, although I'm not insane enough on a bike to be a real challenge. Together with Martin Bravenboer, we still have an Ironman Triathlon to finish. Eelco Dolstra, a living index of Wikipedia [*citation needed*], Rob Vermaas, Dick Eimers, my greetings to you. Last but not least, Armijn Hemel went to great efforts to save the IRC channel from my incorrect spelling.

Words fail to describe the wonderful role that Annette Bieniusa plays in my life. I tried once, but did not come close. Not even with 91 pages of emails. She may appear cute, small and innocent, but in the mean time she stole my heart. She is my greatest source of happiness and support. Although she cleverly managed to keep attribute grammars out of our discussions, she bravely read through my thesis and gave me valuable comments.

Tot slot wil ik mijn ouders en zusjes bedanken. Ik heb dankbaar geprofiteerd van het gemak dat het spreekwoordelijke hotel Middelkoop bood. Ik hoop jullie de komende tijd wat meer aandacht terug te kunnen geven.

Sponsors. To allow me to work on this thesis, I am grateful to Microsoft Research in Cambridge for granting me a Ph. D. Scholarship, which made my employment at the Universiteit Utrecht possible. I was fortunate that the department was willing to finance my fourth year, despite the harsh financial constraints that universities face nowadays. Also, the European project LerNet and the Universidade Federal de Minas Gerais partially financed me so that I could work for half a year in Brazil. Many ideas in this thesis arose during my daily walks to and from the university in Bello Horizonte. Finally, the European project FITTEST allowed me to stay a year longer in Utrecht, so that I could apply attribute grammars to implement bytecode transformations in a fancy way.

Contents

1 Introduction

This thesis investigates the application of attribute grammars to the description of inference algorithms (as implemented in a compiler) that are specified by a collection of inference rules. Such collections are a means to specify the semantics of programming languages. Since there exists no universal inference algorithm for inference rules, it makes the description of inference algorithms a nontrivial exercise. Our goal is to use attribute grammars to make it easier to write such descriptions.

1.1 Overview

Language specifications. A formal programming language specification describes the language's notation and the notation's meaning, and forms a set of requirements for tools that process programs written in this language. A denotational semantics of a programming language is often expressed as a relation between properties of programs written in this language, where the relation is described by a set of inference rules. The structure of such a program is the Abstract Syntax Tree (AST) obtained by parsing the source code of the program, and is one of the properties. For example, in case of code generation, a denotational semantics expresses a relation between the AST and machine code, whereas in case of type checking, the semantics typically relates the AST to a type for each program fragment and an environment with a type for each identifier.

Implementations. A compiler analyzes a program and computes properties of that program. This analysis is specified by a *static semantics*, which describes a relation between programs and properties. For example, the specification of a compiler that translates source code to machine code describes a relation between a program and machine instructions. The properties computed by a compiler can be seen as the evidence that there exists a proof that the properties are related to the program. Assuming that the relations are specified with inference rules, such a proof has the form of an attributed tree. Each node represents an application of an inference rule of the semantics, and attributes describe how the rules are instantiated.

A semantics is *syntax directed* if the shapes of the proofs are isomorphic to the AST. A semantics is *algorithmic* if additionally the constraints between attributes in the tree are expressible as computable functions. As a compiler's implementation is usually based on traversals of the AST, an algorithmic semantics is a convenient specification of a compiler. Since the implementation boils down to traversing the tree and applying the computable functions as mentioned in the specification in the right order, there exists a clear correspondence between the specification and the implementation. An example is a semantics for an assembly language where the relation between the AST and machine code is actually a bijection.

However, most language's semantics are not entirely algorithmic. We come back to this point further below.

Attribute Grammars (AGs) [Knuth, 1968] are an attractive domain specific programming language for the implementation of an algorithmic semantics. Attributes represent inductive properties of the AST. A context-free grammar describes when an AST is correctly structured by relating productions to nodes in the AST. An AG describes when an AST is correctly decorated with attributes by imposing constraints on the attributes per node of the AST, formalized by rules per production. These rules are computable functions between attributes denoted in some host language. AGs abstract from the traversal over the AST and from the order of evaluation of the rules. Therefore, AGs are composable, which makes it possible to describe the implementation of a large language as various separate aspects. Moreover, an AG is compilable to an efficient algorithm that computes the attributes. Thus, an AG serves both as a specification and an implementation of an algorithmic semantics.

Declarative specifications. As argued above, to implement a denotational semantics, it is preferable that the semantics is algorithmic because of the close correspondence with an implementation. A declarative semantics is a semantics suitable for formal reasoning and documentation purposes, and is usually more abstract and concise than an algorithmic semantics.

For example, consider a denotational semantics that describes a relation between the AST and a sequence of machine instructions. When the relation is not functional, there may be many related sequences of machine instructions for a given AST. In particular, there may be a difference between a shortest sequence of instructions, or a sequence of instructions with the lowest expected execution time. The choice of which sequence is computed is left up to the implementation or specified separately.

Moreover, declarative features of a programming language can usually be more concisely described relationally instead of being based on a computable function. An example is operator overloading, where the choice of the implementation of an operator depends on the types of the operands. Effectively, the compiler takes care of some of the work of the programmer, and models the inference of the proof in some way.

From ASTs to proofs. A transformation from a non-algorithmic semantics to an algorithmic semantics is non-trivial. An implementation using traversals of the AST is based on an algorithmic semantics. Consequently, it is difficult to keep the implementation consistent with the declarative language specification.

In this thesis, we approach the implementation of a denotational semantics from another direction. We consider AGs based on the grammar that underlies the inference rules of the relation instead of the language's grammar. As a particular advantage, the AG is closely related to the declarative specification. However, the AG specifies when an attributed proof is valid, but does not specify how to obtain the proof. Since there exists no general procedure that maps such an AG to an algorithm that infers the proof, we need to augment the AG with additional information to obtain such an algorithm.

Thesis. In this thesis, we focus on the description of type inference algorithms with attribute grammars. Most type inference algorithms are a complex combination of a small set of inference techniques, such as type variables and unification to calculate with values that are not fully known yet, fixpoint iteration to approximate solutions, constraints to defer the inference of a subtree, and search tree construction to encode alternative solutions. Such techniques are based on gradually building a proof and exploring intermediate candidate solutions.

In our approach, we conservatively extend AGs to support these techniques and balance between assumptions about the evaluation algorithm, and the preservation of the declarative nature of the description. The central concept that underlies our extensions are higher-order, ordered attribute grammars. In such higher-order AGs, the domain of an attribute can be an attributed tree, which allows us to dynamically grow the tree. The state of a tree is described by a configuration, which specifies the decorations that have been computed. In an ordered AG, the configurations are linearly ordered. A visit, a unit of evaluation for a node, transitions the state of the node to a state described by the next configuration. This concept offers control over the simultaneous evaluation of attributes and exploration of the tree.

Chapter organization. This chapter presents background information and a short outline. We assume that the reader has a strong background in the programming language Haskell [O'Sullivan et al., 2008], is familiar with type systems [Pierce, 2002], and knows the basics of attribute grammars [Knuth, 1968]. This chapter gives (rather) informal definitions of relevant concepts and provides pointers to literature.

The actual contents of the thesis start with the next chapter, Chapter 2. Chapter 2 gives a detailed summary of our extensions and shows how these fit together. Each following chapter covers an extension in detail.

We address concepts of type systems and attribute grammars in combination with notation in Section 1.2 and Section 1.3. In particular, we recast the notion of ordered attribute grammars. Section 1.4 addresses previous work, Section 1.5 gives an overview of each of the extensions, and Section 1.6 sketches the overall goal. Finally, Section 1.7 addresses related work.

1.2 Background on Type Systems

In later chapters, we use type systems based on variants of the lambda calculus as example. In this section, we give a short summary on the lambda calculus, show the evaluation of lambda terms, and give a type system. Furthermore, the discussion of variants of the lambda calculus and their type systems serves as a vehicle to discuss design and implementation challenges of type systems, which we use in the motivation of this thesis in Section 1.5. Another purpose of this section is to introduce vocabulary and notation for subsequent chapters.

We assume that the reader is already familiar with the lambda calculus and type systems. Introductory books on type systems [Pierce, 2002, Harper, 2010] provide a more extensive and formal explanation.

1.2.1 Specification of Programming Languages

Programming languages are described by a grammar since a grammar specifies the set of programs that belong to the language. A semantics relates a program to some properties in a given domain. When these properties represent the correctness of the program, or represent a program in (another) programming language, we talk about a static or denotational semantics. When the properties describe the runtime behavior of the program (typically in the form of state transitions), we talk about a dynamic or operational semantics.

We assume that the specification of a programming language consists of a context-free grammar (Section 1.3.1) and static semantics. The specified programming language is called the *object language*, and a program an *object term*. In case of compilation and transformation, the object language is often referred to as the *source language* and object terms as *source terms*. The language that describes the specification and the language in which the compiler is implemented are called *meta languages*. Finally, in case of a translation to a different language, the language where we translate to is called the *target language* and the translated program a *target term*.

1.2.2 The Lambda Calculus

The lambda calculus is a language that is often used in programming language research, and in research on type systems in particular. Many concepts of programming languages have their roots in variants of the lambda calculus, or have been well-studied in such a context. We discuss the lambda calculus, because we use its concepts in object languages, source languages and target languages in the following chapters.

Figure 1.1 shows the abstract syntax e of expressions in an explicitly-typed variant of the simply-typed lambda calculus, which may contain types τ. In passing, we also give syntax for environments and evaluation contexts. The structure of e is called the Abstract Syntax Tree when represented as a tree (Section 1.3.1).

In the lambda calculus, a function may be passed as a value v in the form of a lambda expression, and thus is *first class*. Such a function is anonymous, although the function can be given an explicit name by binding the function to an identifier. For example, the following expression denotes the application of a function that applies the identity function that it receives as an argument to the value 3:

$$(\lambda y\!:\!I \rightarrow I \,.\, y\,3)\;(\lambda x\!:\!I \,.\, x) \quad \text{-- evaluates to 3 of type } I$$

To specify to what value an expression evaluates, we provide below an operational semantics. In this semantics, the simultaneous substitution of all free x in e_1 by e_2 is denoted by $[x\!:=\!e_2]\;e_1$. The semantics[1] consists of the reduction relation $e_1 \rightsquigarrow e_2$, which is the smallest relation that satisfies the following inference rules[2] of Figure 1.2, which we explain below.

[1] Such a semantics is called a small-step operational semantics because the inference rules describe a transformation step from an intermediate term to another intermediate term, and the actual transformation is the exhaustive application of these transformation steps.

[2] In Section 1.2.3 we will actually consider a notation for inference rules and their interpretation.

$e ::= e_1\ e_2$	-- application, with expression e_1 and e_2
$\mid\ x$	-- variables, e.g. x, y, z
$\mid\ v$	-- values (head-normal form)
$v ::= i$	-- integer constant, e.g. 3 and 42
$\mid\ \lambda x : \tau . e$	-- abstraction, with param x of type τ and body e
$\tau ::= I$	-- integer type
$\mid\ \tau_1 \to \tau_2$	-- function type
$\Gamma ::= \emptyset$	-- empty environment
$\mid\ \Gamma, x : \tau$	-- environment Γ, with on top a mapping of x to type τ
x, y, z	-- variables
f, g, h, a, e	-- expressions

Figure 1.1: Syntax of the explicitly-typed lambda calculus variant.

$$\boxed{e_1 \rightsquigarrow e_2}$$

$$(\lambda x : \tau \,.\, e_1)\ e_2 \rightsquigarrow [x := e_2]\ e_1 \ \textsc{beta} \qquad \frac{e_1 \rightsquigarrow e_2}{e_1\ e \rightsquigarrow e_2\ e}\ \textsc{left}$$

Figure 1.2: Operational semantics as a reduction relation on expressions e.

A *beta-redex* is an expression of the form $(\lambda x.e)\ a$ which can be reduced. The above rules describe normal-order reduction. Through the rule LEFT the beta-redex in the head position is identified. Indeed, if we consider the reduction of the above example, we end up first substituting f with the identity function, such that we obtain $(\lambda x : I \,.\, x)$ 3, and then substitute 3 for x, such that we end up with the value 3. When no reductions are possible anymore, the expression is in *head normal form*.

1.2.3 Type Rules

The purpose of a static semantics is to exclude programs that incorrectly use their data. A type system classifies expressions as well-typed if it can associate a type with it. A type system is sound if it has the subject reduction property, which means that after each reduction step, the resulting term has the same type as the original expression. Type systems typically ensure the absence of certain programming errors, such as passing an integer where a function is expected.

As an example, we give a type system specification for the lambda calculus as defined above in Figure 1.4 and explain it below. We allow liberal syntactic sugar in our specifica-

$r ::= \forall \overline{x} \,.\, d$	-- quantified rule over meta variables $\overline{x}$
$d ::= j_1 \,...\, j_n \,;\, j$	-- rule with premisses $\overline{j}$, conclusion j, $n \geqslant 0$
$j ::= R^n\, \overline{m}$	-- judgment with $\lvert\overline{m}\rvert = n$
R^n	-- n-place relation R (n often omitted)
m	-- argument (meta expression)
x, y	-- meta variables for values in the object language

The notation $\frac{j_1 \,...\, j_n}{j}$ (with optional rule label) is sugar for $j_1 \,...\, j_n \,;\, j$. The explicit equality $x_1 \approx x_2$ denotes a judgment $Eq^2_\tau\; x_1\; x_2$, with relation Eq_τ representing structural equality of object terms of meta-type τ. The quantification of meta variables is usually left implicit.

Figure 1.3: Notation for inference rules.

$$\boxed{\Gamma \vdash e : \tau}$$

$$\Gamma \vdash i : I \;\textsc{con} \qquad \frac{(x:\tau) \in \Gamma}{\Gamma \vdash x : \tau}\,\textsc{var} \qquad \frac{\Gamma \vdash f : \tau_1 \to \tau_2 \quad \Gamma \vdash a : \tau_1}{\Gamma \vdash f\; a : \tau_2}\,\textsc{app} \qquad \frac{\Gamma, x : \tau_1 \vdash e : \tau_2}{\Gamma \vdash \lambda(x:\tau_1).e : \tau_1 \to \tau_2}\,\textsc{lam}$$

Figure 1.4: Type rules of the explicitly typed lambda calculus variant.

tions. Depending on the context, the notation $\overline{x}$ represents a sequence or set $x_1, ..., x_n$.

We define a typing relation $\Gamma \vdash e : \tau$, which is notation for a three-place relation over Γ, e, and τ. A judgment of this relation states that in environment Γ, the expression e has type τ. We typically describe such relations with inference rules (type rules). Figure 1.3 gives the syntax of inference rules. A type rule r consists of zero or more premisses (judgements $\overline{j}$) above the line, and a conclusion j below the line. We come back to meta expressions m later, but it contains at least the meta variables x.

Figure 1.4 shows the rules of the lambda calculus as introduced above. The rule CON associates the type I to any integer constant. The VAR rule associates the type to the identifier as it is bound to that identifier in the environment. The APP rule requires expression f to be a function that takes an argument of the same type as its formal parameter. The type associated with the application itself is the result type of the function. In LAM, the body of the lambda may assume a type τ_1 associated with identifier x in the environment.

The typing relation is the smallest relation that satisfies a set of rules r, which serve as axioms of the relation. In a judgment, meta expressions m are arguments to some relation R. The language of meta expressions is a formal language, which is usually left implicit, but facilitates the construction of symbolic and concrete object terms.

A Relation R can be a built-in relation (an *atomic* relation) or relation described by inference rules. Instead of using expressions m to construct symbolic terms, atomic relations for each production in the object language can equivalently be used to constrain symbolic object terms. For example, for the arrow-production in the syntax of types, we assume the existence of an atomic relation $C_{\rightarrow}\ \tau_1\ \tau_2\ \tau_3$ that expresses $\tau_1 \equiv \tau_2 \rightarrow \tau_3$. Then, instead of a judgement $R\ (\tau_2 \rightarrow \tau_3)$, we may write $R\ \tau_1$ with a fresh τ_1 and $C_{\rightarrow}\ \tau_1\ \tau_2\ \tau_3$.

The rules are in canonical form when the arguments of relations in the judgments consist only of meta variables, with the single exception of the judgment denoted $m \rightsquigarrow^{*} x$ which represents a reduction relation on object terms where m is a meta expression. Additionally, each meta variable occurs at most once, not counting its occurrences in (additional) explicit equalities. We assume in this thesis that meta expressions are written in Haskell. For reasons of simplification we assume below that meta expressions in this chapter are written in the above lambda calculus so that the definitions of m and e coincide. To rewrite rules into canonical form we use the operational semantics of (meta) expressions. For example, we replace a judgment $R\ e$ to $R\ x$ and introduce the additional premise that $e \rightsquigarrow^{*} x$, where x is fresh and $\rightsquigarrow^{*}$ is the exhaustive application of the reduction relation on (meta) expressions.

To deal with the occurrences of meta variables, we introduce additional explicit equalities and quantified fresh variables. Rules in canonical form are more verbose, but easier to formally reason with.

Given arguments $\overline{a}$ for the typing relation R described by inference rules, we can prove that these are a member of the typing relation, denoted by $R\ \overline{a}$ or $\overline{a} \in R$, by constructing a *derivation tree* using the rules of the relation. A derivation tree is a proof for $R\ \overline{a}$ when there is a rule of R with a conclusion that matches against $R\ \overline{a}$ (with substitution θ), and for each premise of the rule (with θ applied), there is subtree that is a proof for that premise. Proofs of atomic relations are leafs of a derivation. A derivation tree for the earlier example is:

$$\dfrac{\textsc{lam}\ \dfrac{\dfrac{\textsc{var}\ \dfrac{(y : I \rightarrow I) \in \Gamma_1}{\Gamma_1 \vdash y : I \rightarrow I} \qquad \Gamma_1 \vdash 3 : I\ \textsc{con}}{\Gamma_1 \vdash y\ 3 : I}\ \textsc{app}}{\Gamma_0 \vdash \lambda y : I \rightarrow I\ .f\ 3 : (I \rightarrow I) \rightarrow I} \qquad \dfrac{\dfrac{(x : I) \in \Gamma_2}{\Gamma_2 \vdash x : I}\ \textsc{var}}{\Gamma_0 \vdash \lambda (x : I).x : I \rightarrow I}\ \textsc{lam}}{\Gamma_0 \vdash (\lambda y : I \rightarrow I\ .f\ 3)\ (\lambda x : I\ .x) : I}\ \textsc{app}$$

$\Gamma_0 = \emptyset$	-- initial environment
$\Gamma_1 = \Gamma_0, y : I \rightarrow I$	-- extension of Γ_0 in the application of LAM (left branch)
$\Gamma_2 = \Gamma_0, x : I$	-- extension of Γ_0 in the application of LAM (right branch)

Each node v in the derivation tree is associated with a judgment $j_v = R_v\ \overline{a}$, and the subtree rooted by v is a proof that $\overline{a} \in R_v$, where R_v is the associated relation. Furthermore, node v is associated with some rule r_v, which determines the structure of v. The node is furthermore decorated with values for the meta variables that are bound by r_v.

Type rules are *syntax-directed* when in derivation trees, for each node v, the associated rule r_v is uniquely determined by an argument a (which represent the expression for which we try to construct the proof) of the associated judgment j_v. Thus, for syntax directed rules, the choice of what rule to apply in the construction of the proof is determined uniquely by productions of the object language, and there is a one-to-one mapping between nodes in the abstract syntax tree and nodes of the derivation tree. This is a desirable property because it means that the shape of the derivation tree depends only on the structure of the object term, and is thus given at the start of the proof construction: we know what rules to apply where, and are only left with the problem of how to make their instances match.

However, when the correspondence between r_v and the a priori known arguments $\overline{i} \subseteq \overline{a}$ of j_v is not functional, we call the rules *declarative*[3]. Additionally, when r_v also depends on the inferred type τ, with $\tau \in \overline{a}$, the rules are declarative and also *type directed*.

A rule r_v imposes two forms of constraints on node v in the derivation tree. Constraints on the structure (as discussed above), and constraints on the values for the node's meta variables that decorate v (as discussed below). When constructing a proof, we choose rules and values that satisfy these constraints. We call the description of such choices *aspects* of the rules. These aspects are declarative when the decisions are not functionally determined by $\overline{i}$. Declarative aspects complicate the construction of a proof, because there may be many choices that seem suitable to complement a partial proof, but turn out later to be inappropriate. In Section 1.2.4, we discuss strategies to resolve such declarative aspects.

We determine which meta variables are declaratively defined with a transformation of the relations into functions: those meta variables that make the function non-deterministic are declaratively defined. For that, each parameter of a relation must have a specification that declares it (conditionally) as either an input or output. In a judgement $j = R\,\overline{x}$ of a rule in canonical form, a meta variable $x \in \overline{x}$ is at an input position when j is a premise and x is passed as output argument, or when j is a conclusion and x is passed as an input argument. Otherwise, a meta variable is at an output position. In case of explicit equalities $x_1 \approx x_2$, x_1 and x_2 are both at an input position. In case of the reduction relation $m \rightsquigarrow x$, the occurrences of meta variables of m are at an input position and x is at an output position.

In case of the construction/destruction relation C_P related to the production P of the object language, in the judgment $C_P\; x_0\; x_1 \ldots x_n$, $x_1, \ldots, x_n$ are at an input position if and only if x_0 is at an input position.

A variable is *declarative* if none of its occurrences are on an output position and the variable does not occur in an explicit equality with a non-declarative variable.

As an example, we add a type rule for an implicitly typed lambda abstraction, and discuss which meta variables are declarative. As preparation we show the canonical forms LAM-IMPL' and APP' in Figure 1.5. When we assume that types are an output of the relation, the meta variable τ_1 of LAM-IMPL is declarative. When we assume that types are an input of the relation, the meta variable τ_1 of APP is declarative.

The above sketch of an analysis assumes that the atomic relations are computable functions. We call type rules declarative if they exhibit declarative aspects, and *algorithmic* otherwise. In the latter case, when the rules are described in Ruler (Section 1.4), a type inference algorithm can be generated.

1.2.4 Type Inference

A type checking or type inferencing algorithm concerns itself with constructing derivation trees when given the main judgment. In case of type checking the type is part of these a

[3] Some authors call declarative rules *nondeterministic* because an inference algorithm may need to choose nondeterministically what rule to apply.

$$\frac{\Gamma, x : \tau_1 \vdash e : \tau_2}{\Gamma \vdash \lambda x.e : \tau_1 \to \tau_2}\ \textsc{lam-impl} \qquad \frac{\Gamma \vdash f : \tau_1 \to \tau_2 \qquad \Gamma \vdash a : \tau_1}{\Gamma \vdash f\ a : \tau_2}\ \textsc{app}$$

$$\frac{\begin{array}{c}\Gamma, x : \tau_1 \vdash e : \tau_2 \\ C_{\to}\ m_2\ \tau_1\ \tau_2 \qquad C_{App}\ m_1\ x\ e\end{array}}{\Gamma \vdash m_1 : m_2}\ \textsc{lam-impl}' \qquad \frac{\begin{array}{c}\Gamma \vdash f : m_2 \qquad \Gamma \vdash a : \tau_1 \\ C_{\to}\ m_2\ \tau_1\ \tau_2 \qquad C_{App}\ m_1\ f\ a\end{array}}{\Gamma \vdash m_1 : \tau_2}\ \textsc{app}'$$

Figure 1.5: Rules LAM-IMPL and APP, and their respective canonical form.

priori known arguments. In case of type inference the type is not, and actually inferred. This distinction between type checking and type inference is rather vague, because a type system may exhibit other declarative aspects than the definition of types, and thus implies some form of inference. Moreover, for a compiler typically the inferred arguments matter but the derivation tree is not of direct interest.

Inference as a forest of derivation trees. Some authors consider type inference as a two-step process: a traversal of the AST to generate a set of constraints, and solving this set of constraints. We shall, however, consider type inference as the incremental construction of a forest of derivation trees, which has a closer connection to declarative specifications. Its intermediate state is a forest of partial derivation trees. It starts with a singleton forest containing an empty derivation tree with the main judgment as pending aspect. A derivation tree in the forest has an associated status, which is that it is partial, complete, or unsatisfiable.

A reduction step in this forest consists of cloning an existing partial derivation and adding the cloned derivation to the forest after resolving one pending aspect of the clone such that the number of unique derivation trees in the forest increases.

An aspect is either a meta variable or a judgment. To resolve the former, the meta variable is bound to a value in its domain. To resolve the latter, it depends on whether the relation of the judgment is described by rules or an atomic relation. If the relation is described by rules, either a rule can be applied which leads to a larger or a complete derivation tree, or the derivation tree is unsatisfiable. If the relation is atomic, then the aspect is solved by running an algorithm that is associated to the relation which either succeeds or fails.

The existence of a complete derivation means that the source program is type correct. When all reductions have been applied exhaustively and there exists no complete derivation the program is type incorrect. However, there may be infinitely many partial derivations. Also, there may be infinitely many partial derivations of a certain height, for example, when the domain of a meta variable is infinite. We come back to this issue later: in practice, an inference algorithm uses a less general approach.

Naive algorithm. The algorithm in Figure 1.6 represents a typical Prolog-style strategy for the construction of derivation trees. We explain some aspects of the example below.

```
$prove :: j \to Inf\ Tree$                 -- $Inf$ offers unification and backtracking
$prove\ (x \approx y) = \mathbf{do}$       -- case for explicit equality
  $unify\ x\ y$                            -- equality mapped to unification
  $return\ (x \sim y)$                     -- builds equality node
$prove\ (R\ \overline{x})$                 -- case for judgments
  $\mid atomic\ R\ \ = extern\ R\ \overline{x}$   -- proven externally
  $\mid otherwise = \mathbf{do}$                 -- relation specified by rules
    $r\ \ \ \ \ \ \ \ \leftarrow rules\ R$       -- picks a rule (backtrack point)
    $(ps\ ;\ c) \leftarrow instantiate\ r$       -- instantiate rule
    $zipWithM\ unify\ (args\ c)\ \overline{x}$   -- bind args
    $ts \leftarrow mapM\ prove\ ps$              -- recursion on premises
    $return\ (ts \rhd_{name\ r}\ c)$             -- build derivation node
```

Figure 1.6: Sketch of a general inference algorithm.

In this example, we refer to an inference monad *Inf* which takes care of backtracking and unification. Rules are tried in a predefined order. If a rule cannot be applied, *backtracking* occurs to the next rule.

Meta variables are initially represented as symbolic values, and unified with other meta variables or object terms during inference. Unification is derived from an equivalence relation, which can in turn be derived generically from algebraic data type declarations.

The operation *rules* introduces a backtrack-point. It tries the rules in a given order by feeding the rules one by one to the continuation. A later unification may fail and cause a backtracking to that point. The operation *instantiate* substitutes the quantified meta variables with fresh meta variables. The instruction *extern* proves a judgment externally and returns evidence for it if it succeeds. With combinators $\rhd$ (internal nodes) and $\sim$ (equality leafs) we construct evidence in the form of a derivation tree.

Undesirable properties of the algorithm. The above algorithm is incomplete and does not terminate for all but the simplest type systems. This is necessarily the case because there exist type systems for which inference is undecidable. Via backtracking, only finite and inductively defined object terms can be inferred, and unifications only produce compositions of object terms. The algorithm is also inefficient. The order in which rules are tried may cause poor performance or nontermination. Moreover, the order in which rules are tried is not specified, which gives unpredictable results. Although the results are sound, these may not be optimal.

In practice, typical algorithms refrain from backtracking or constructing many candidate derivations, because the search space is too large when dealing with ASTs with thousands of nodes. Therefore, actual inference algorithms resemble this overall approach, but select rules and instantiations of meta variables in a more sophisticated way.

Actual inference algorithms. Actual inference algorithms apply various strategies (depending on the form of the declaratie aspect) to get around the above undesirable properties.

Unification can be used to deal with equivalence judgments in type rules. Alternatively, an algorithm may collect several candidate constraints on a meta variable, then pick an object term that is the least solution to all these constraints. When constraints are monotonic, a value can gradually be approximated via fixpoint iteration with an initial bottom value. Type and effect systems often require such algorithms.

When rule selection is declarative, we typically want the choice to be a function of the syntax of the language (*syntax directed* rules) or other object terms (e.g. *type directed* rules). Alternatively, some rules can be restricted to only be applied in a proof after other rules have been applied so that the choices between the remaining rules becomes functional in the above sense. This is not always possible: the applicability of a rule may depend on unresolved meta variables and may require the rule selection to be deferred which is called residuation Hanus [1994].

Challenge: orchestration of strategies. The orchestration of such strategies is a complex undertaking. The order in which strategies are applied may influence the result, and it may not always be clear when to start or stop applying strategies. These are all challenges an implementation must deal with.

Challenge: annotations. The holy grail of type system research is to define expressive type systems (for a class of programs) that have sound and complete (decidable) inference algorithms. Given two type systems (that satisfy type-soundness with respect to the operational semantics of the language), one type system is more expressive than another type system if accepts a superset of the programs that the other accepts.

To bypass the strict undecidability boundaries, many languages allow programmers to assist the inference progress by providing additional information (e.g. type annotations) in the object program that translates to concrete bindings for otherwise declarative meta variables. There are delicate balances between expressiveness of the type system, the amount of annotation to be provided by the programmer, and the predictability of inference.

For example, the Damas-Hindley-Milner type system and accompanying inference algorithm (Section 1.2.6) does not require any annotations to help the inference process, but disallows functions with parameters of a polymorphic type. System F, the polymorphic lambda calculus, in comparison allows funtions with a polymorphic type, but requires an abundance of type annotations. As a middle way, HML [Leijen, 2009] expresses all of System F, but requires only type annotations for polymorphic lambda parameters. FPH [Vytiniotis et al., 2008] positions itself in between DHM and HML. It requires a type annotation when applying a function to an argument with a polymorhic type. Dijkstra and Swierstra [2006a] proposed a global flow analysis that propagates type annotations to locations where (a part of) the annotation is also applicable.

Challenge: type errors. As stated before, type inference concerns itself with the construction of derivation trees. If no such derivation exists, it is considered a type error. Type rules only specify under what condition a type is acceptable. Thus, an implementation may require additional information to produce understandable error messages [Heeren et al., 2003a].

Challenge: multiple derivation trees. A partial order on types specifies when one type is more general than another type. A type is *principal* when it is the most general type of an expression. For reasons of predictability and modularity it is desirable that a principal type exists for an expression and that an algorithm infers a derivation for this type. This notion of principality can be extended to derivations [Jim, 1996].

For the above type rules, there are infinitely many derivation trees for $(\lambda x.x)$, but none of the accompanying types (e.g. $\tau \rightarrow \tau$ for any type τ) are comparable, thus this expression has no principal type (thus also not a principal derivation) for the given type system.

There are several remedies. The type system can be changed such that more type annotations need to be given, or that choices can be deferred by encoding pending choices in the type language as constraints. Polymorphism and qualified types are an example of the latter: type schemes are introduced to represent a delayed choice of types, as we show in the next section.

1.2.5 Parametric Polymorphism

For the identity-function $\lambda x.x$, there are no constraints on the type of x. During inference, no binding arises for the meta variable associated to x. We can thus bind a fresh type constant α to the meta variable, which leads to the type $\alpha \rightarrow \alpha$ for $\lambda x.x$. To specify that α can be any type, we universally quantify over α, and obtain the type $\forall \alpha.\alpha \rightarrow \alpha$. This process is called abstraction or *generalization*. A polymorphic type represents many types which can be obtained by *instantiating* the quantified type constants (*type variables*). Moreover, we obtain a proof for the instantiated type by instantiating the proof for the generalized type in an analogous way.

Polymorphic types enable *parametric polymorphism*, which allows functions to be called with parameters of different but acceptable types. Parametric polymorphism is important for the use of Haskell's many convenient higher-order functions, such as *id*, *flip* and \$.

Quantified types. The *Damas-Hindley-Milner* (DHM) type system serves as the classical example of a type system with polymorphic types. It forms the basis of the type systems of ML and Haskell, and underlies the type systems of many other languages. DHM's type language is an extension to that of the simply typed language calculus. It additionally contains type variables α and poly types (*type schemes*) σ:

$\tau ::= \alpha$	-- type variable
$\quad \mid \ \tau_1 \rightarrow \tau_2$	-- function type
$\sigma ::= \forall \alpha.\sigma$	-- poly type: may have a qualifier
$\quad \mid \ \tau$	-- mono type: has no quantifiers
α, β	-- type variables

$$\boxed{\Gamma \vdash e : \sigma}$$

$$\frac{(x:\sigma) \in \Gamma}{\Gamma \vdash x : \sigma}\ \textsc{var} \qquad \frac{\Gamma \vdash f : \tau_1 \to \tau_2 \quad \Gamma \vdash a : \tau_1}{\Gamma \vdash f\ a : \tau_2}\ \textsc{app} \qquad \frac{\Gamma, x : \tau_1 \vdash e : \tau_2}{\Gamma \vdash \lambda x.e : \tau_1 \to \tau_2}\ \textsc{lam}$$

$$\frac{\Gamma \vdash e : \sigma \quad \Gamma, x : \sigma \vdash b : \tau}{\Gamma \vdash \mathbf{let}\, x = e\, \mathbf{in}\, b : \tau}\ \textsc{let} \qquad \frac{\Gamma \vdash e : \sigma \quad \alpha \notin \mathit{ftv}\, \Gamma}{\Gamma \vdash e : \forall \alpha.\sigma}\ \textsc{gen} \qquad \frac{\Gamma \vdash e : \forall \alpha.\sigma}{\Gamma \vdash e : \sigma\, [\alpha := \tau]}\ \textsc{inst}$$

Figure 1.7: The DHM type system.

A mono type τ does not contain universal quantifiers. A poly type can be interpreted as the infinite set of mono types, where each quantified type is substituted by some mono type. The identity function $\lambda x.x$ has the type $\tau \to \tau$ for any type τ. It can thus be given the poly type $\forall \alpha.\alpha \to \alpha$. The type variable α in such a type is an object term that is not to be confused with a meta variable.

The distinction between mono and poly types is important in the declarative type rules of DHM. Lambda abstractions take mono types as parameter[4], and have a mono type as result. Polymorphic types are introduced in the environment by (non-recursive) let-expressions:

$$\begin{array}{ll} e ::= \ldots & \\ \quad \mid\ \mathbf{let}\, x = e_1\, \mathbf{in}\, e_2 & \text{-- generalized type of } e_1 \text{ is visible as } x \text{ in } e_2 \end{array}$$

Although a let-expression can be interpreted as $(\lambda x.e_2)\ e_1$, the typing derivation may differ depending on the presence and application of a generalization rule.

The DHM type system in Figure 1.7 consists of the following rules of which we mention some details below. Via one or more applications of the INST rule, poly types may be instantiated to mono types by replacing a bound type variable with a mono type. Conversely, via rule GEN a poly type can be constructed using quantification over a type variable, provided that the substitution does not capture the free type variables in the environment.

Qualified types. In general, meta variables may be constrained by judgments that by themselves are insufficient to bind a concrete type to a meta variable. As an example, the following type rules encode overloading of the addition operator on both integers and floats.

$$\Gamma \vdash \_+\_ : I \to I \to I\ \textsc{add.int} \qquad \Gamma \vdash \_+\_ : F \to F \to F\ \textsc{add.float}$$

[4] It has been shown that the inference of a polymorphic type for a parameter of a recursive function is undecidable in general [Wells, 1999]. Many type systems thus impose restrictions on the types of function parameters to make inference feasible.

In the expression $\lambda x.x+x$, the type rules for addition constrain x to be a numeric type, but do not dictate whether this type is an integer or a float. At other locations in the derivation tree there may be constraints imposed on the type that make it clear which of the rules applies.

During type inference, the type may be insufficiently constrained to resolve the judgment. A typical strategy is to defer the judgment until the end of the inference of the scope in which the judgment arose, which is usually at generalization points. If the type is still not constrained sufficiently, a strategy is to *default* to one of the applicable rules. Another strategy is to encode the judgment as part of the type (if potentially satisfiable) and delay the judgment to all locations where the generalized type is instantiated. Such an encoded judgment is called a qualifier.

A qualified type of the expression $\lambda x.x+x$ given the above rules is:

$$(\lambda x.x+x) :: \forall \alpha\, \beta . \exists \gamma . (\gamma \vdash \_+\_ : \alpha \to \alpha \to \beta) \Rightarrow \alpha \to \beta$$

With additional information this type can be refined. An equivalence between β and α is deducable from the rules ADD.INT and ADD.FLOAT. Also, suppose that we know further that a qualifier $\gamma \vdash \_+\_ : \alpha \to \alpha \to \alpha$ is simplifyable to a qualifier $\mathit{Num}\ \alpha$, the type is refineable to:

$$(\lambda x.x+x) :: \forall a . \mathit{Num}\ a \Rightarrow a \to a$$

Haskell's overloading with type classes actually brings such reasoning under the control of the programmer.

Such extensions to a type system do not come for free. When the code generation depends on the proof of a deferred judgment, the generated code needs to be parametrized by information that is derived from the deferred proof. Moreover, a function may be given a type with qualifiers that can never be satisfied, which we may only find out when we try to use an identifier with such a type. Some type systems define a coherence relation on qualifiers to formalize the potential satisfaction of constraints.

1.2.6 Damas-Hindley-Milner Inference

In this section we consider type inference for the DHM type system. The naive backtracking approach as presented earlier may easily lead to nontermination, because the INST and GEN rule can be alternated as each others inverse indefinitely. However, the syntactical restrictions on types permits a more appropriate inference strategy.

With some effort, we can deduce that generalization has only an effect for the toplevel expression, or for the expression e in a let-expression. It is also sufficient to perform instantiation only after taking the type of an identifier from the environment in the rule VAR. According to the interpretation of poly types, a type quantified over a variable that does not occur free in the type describes actually the same set of mono types as the type without the quantification, thus we only need to consider types that occur free in the type as generalization candidates. Moreover, the order of the type variables over which is quantified is irrelevant.

Inference algorithm W exploits these properties. It is a sound and complete implementation of the rules, produces most general types, and needs to examine each node of the AST only

In the description of algorithm W we assume monadic versions of the above functions to form an inference monad *Inf* for the implementation of the DHM type system. Essentially, these monadic functions are wrappers around the above functions:

```
type Inf = RWST Env Subst Errs     -- reader, writer, and state monad
fresh_I ::                Inf Ty       -- returns a fresh type
unify_I :: Ty → Ty → Inf ()          -- unifies two types
gen_I   :: Ty      → Inf Scheme     -- generalizes a type to a scheme
inst_I  :: Scheme  → Inf Ty         -- instantiates a scheme to a type
```

The threading of the substitution and collection of error messages is hidden in the monad, as well as the top-down distribution of the environment, so that we define algorithm W as a function $\mathcal{W} :: Expr \to Inf\ Ty$:

```
W ⟨x⟩   = asks (lookup x) >>= inst_I        W ⟨λx.e⟩ = do
W ⟨f a⟩ = do                                   t ← fresh_I
   r ← fresh_I                                 r ← local (insert x t) (W e)
   t ← W f                                     return⟨t → r⟩
   s ← W a                                  W ⟨let x = e in b⟩ = do
   unify_I t⟨s → r⟩                            t ← W e >>= gen_I
   return r                                    local (insert x t) (W b)
```

The notation $\langle e \rangle$ represents the abstract syntax of an object term e. We use this notation to conveniently pattern match and construct object terms in the host language. Note that $\langle x \rangle$ is an AST of the type *Expr* and x is an AST of the type *Identifier*, but that $\langle f\ a \rangle, f$ and a represent ASTs of the type *Expr*.

Figure 1.8: Algorithm W.

once. Many actual inference algorithms are based on algorithm W. We describe algorithm W as a monadic function $\mathcal{W} :: Expr \to Inf\ Ty$ in Figure 1.8.

Unification plays an important role in algorithm W. In this thesis, we assume the following functions have an efficient implementation [Dijkstra et al., 2008]:

```
fresh       ::              Subst → (Ty, Subst)      -- returns a fresh type var
unify       :: Ty    → Ty → Subst → (Errs, Subst)    -- unify two types (improves subst)
generalize  :: Env → Ty → Subst → Scheme             -- generalize a type in a certain env
instantiate :: Scheme     → Subst → (Ty, Subst)      -- instantiate a scheme freshly
```

The type *Subst* represents a substitution and a fresh variable supply, which is a mapping from meta variable to a concrete type. The types *Ty* and *Scheme* coincide with τ and σ respectively. An environment of type *Env* contains bindings from identifiers to types, and *Errs* is the type of a collection of error messages. Similarly, we assume that there is a type *Expr* and *Identifier* that coincide with the nonterminals e and x respectively.

Figure 1.8 shows Algorithm W, which is a recursive traversal of the *Expr* AST. The combi-

```
$\mathcal{W}\ \langle \mathbf{let}\ x = e\ \mathbf{in}\ b\rangle = \mathbf{do}$
    $t \leftarrow \mathit{fresh}_I$                                   -- start with fresh type for binding
    $\mathit{local}\ (\mathit{insert}\ x\ t)\ (\mathcal{W}\ e \gg\!= \mathit{unify}_I\ t)$    -- infer type and match against binding
    $s \leftarrow \mathit{gen}_I\ t$                                   -- generalize over unbound variables
    $\mathit{local}\ (\mathit{insert}\ x\ s)\ (\mathcal{W}\ b)$                    -- infer with generalized scheme in env
```

Figure 1.9: Algorithm W with a recursive let-binding.

nator *local* applies changes to the environment that are only visible in the monadic computation it encapsulates. The function *insert* adds a binding into the environment, which possibly shadows an already existing binding. The *asks* operator exposes the hidden environment, so that we can use *lookup* to recover the type to which an identifier is bound.

The above code assumes a non-recursive let binding. Figure 1.9 shows the algorithm for a recursive let binding. The unification at the end of inference for e binds t to its actual type. All occurrences of x in its own right-hand side must agree with t. To prevent having to deal with polymorphic recursion, x has a mono-type inside its own binding. The generalized type is only available in the body of the let-expression.

The positioning of *local* and gen_I is tricky. Since generalization is performed with respect to free type variables in the environment, the generalization needs to be positioned outside the local environments of the e and b subexpressions, because the type to generalize occurs in these environments.

The monadic formulation of algorithm W is concise because cumbersome flows of environments and substitutions are encapsulated by the monad. However, the abstraction offered by the monad is not always obvious when the object language is more complex. When the object language has pattern-bindings, new bindings arise when visiting the pattern, which means that behavior for the environment is more complex than simply top-down. If multiple derivations are possible, then substitutions may need to be duplicated and merged. The sequencing of operations on the encapsulated state is therefore important.

1.2.7 Polymorphic Lambda Calculus

When we eliminate the distinction between mono and poly types, and thus allow poly types everywhere, we obtain an implicitly typed version of the *System F* type system. It allows expressions such as $\lambda f.g\ (f\ (\lambda x.x))\ (f\ 3)$, where a lambda parameter is applied to values of different types, which is not expressible in the DHM type system. Unfortunately, inference for this system is undecidable [Wells, 1999], and it does not have most general types.

For System F itself, type checking is decidable. However, the syntax is verbose, as type abstraction and type instantiation are explicitly encoded, and types need to be given explicitly for lambda parameters. Figure 1.10 shows the inference rules of System F, and we discuss some aspects of the rules below.

A type application $f\ \sigma$ requires f to have a universally quantified type, and provides the

$$\boxed{\Gamma \vdash e : \sigma}$$

$$\frac{(x:\sigma) \in \Gamma}{\Gamma \vdash x:\sigma}\ \textsc{var} \qquad \frac{\Gamma \vdash f:\sigma_1 \to \sigma_2 \quad \Gamma \vdash a:\sigma_1}{\Gamma \vdash f\ a:\sigma_2}\ \textsc{app.e} \qquad \frac{\Gamma \vdash f:\forall\alpha.\sigma_2}{\Gamma \vdash f\ \sigma_1:\sigma_2\,[\alpha:=\sigma_1]}\ \textsc{app.ty}$$

$$\frac{\Gamma, x:\sigma_1 \vdash e:\sigma_2}{\Gamma \vdash \lambda x.e:\sigma_1 \to \sigma_2}\ \textsc{abs.e} \qquad \frac{\Gamma \vdash e:\sigma \qquad \alpha \notin \mathit{ftv}\,\Gamma}{\Gamma \vdash \Lambda\alpha.e:\forall\alpha.\sigma}\ \textsc{abs.ty}$$

Figure 1.10: System F.

type σ to instantiate this type with. A type abstraction $\Lambda\alpha.e$ quantifies the type of e over α. The syntax of f thus dictates when to apply rule APP.TY and rule ABS.TY.

Variants of System F are typically used for typed backends of compilers, because of its expressiveness. For example, evidence translation of Haskell type classes may need System F types [Faxén, 2002]. As frontend, type systems are proposed that are more restrictive than System F, but more liberal than the DHM type system. It is desirable that such systems do not require type annotations for programs that are acceptable by DHM, and allow for a predictable inference. An inference algorithm then infers a type for such programs, and maps these to System F terms.

1.2.8 Discussion

The DHM inference algorithm in Section 1.2.6 shows that the mapping between a declarative type system specification and its accompanying inference algorithm is not straightforward. The gap between a complex type system and a sophisticated inference algorithm is even larger. As mentioned above, one of reasons is that an inference algorithm requires information that is not present in type rules. Also, type derivations are typically only treated as a model. Inference algorithms compute types from abstract syntax trees, with the underlying assumption that the combination of type and AST can be turned into a derivation tree.

Generalization and instantiation, for example, are often dealt with as part of the unification algorithm in order to support on-demand *impredicative instantiation*, which is the instantiation of a type variable with a poly type. In the type rules these are encoded as separate expression rules.

A declarative specification is typically a minimalistic lambda calculus to explain particular language features, whereas programming languages are much richer in syntax and language features. Algebraic data types are typically not present in declarative specifications, and neither are mutually recursive let bindings, because these are often regarded as syntactic sugar. Actual programming languages, however, require the presence of such features.

We can improve the resemblance between the inference algorithm and declarative specification when the structure of the inference algorithm can be derived from the declarative specification. In this thesis, we propose the use of attribute grammars as the basis for such algorithms. Section 1.3 gives a short introduction.

1.3 Background on Attribute Grammars

In Section 1.2, we followed the common practice of specifying the semantics of programming languages via relations between properties, where the relations are defined by inference rules and the properties include a source term. *Attribute Grammars* (AGs) [Knuth, 1968] are an alternative approach. An AG specifies the semantics of a language as attributes on nonterminal symbols in the grammar of the language.

An AG is a context-free grammar extended with *attributes* and *rules*. We give a definition in Section 1.3.1, in which we describe that attributes are associated with nonterminals, and rules with productions. Given a value for each attribute associated with the nonterminal symbols of a production, the rules specify whether these values are correct. On the other hand, the rules can also be used to compute such attributions for a given AST, which we address in Section 1.3.4.

In the other subsections we describe common extensions, features and uses of attribute grammars for the purpose of showing why AGs are an attractive language for describing the implementation of compilers, but also to give some background information to which we refer from later chapters of this thesis. A shallow scan through these subsections may be beneficial to the understanding of the other chapters of this thesis.

Section 1.7.5 gives an overview of various systems that provide AGs in various flavors.

1.3.1 Syntax of Context-Free Grammars and Attribute Grammars

We introduce a notation for attribute grammars. In this section, we use the term host language to refer to the language in which the algorithm or compiler is written which we generate from the attribute grammar description. As we see later, functions of the host language may appear in grammar descriptions.

We are slightly more formal in this section in comparison to the other sections because in later chapters we introduce various notations for attribute grammars and extensions. Since concepts such as nonterminals, productions, attributes and abstract syntax trees are common to those notations, we introduce these here — although we actually expect that the reader is already familiar with these definitions.

Context-free grammars. Formally, grammars specify languages, but we also use grammars to describe the structure of tree-like data structures. Chomsky [1956] described several classes of grammars with increasing expressiveness and implementation complexity. The class of context-free grammars is particularly convenient for the description of the structure of terms, but is usually not expressive enough to describe the desired correctness properties of programs such as well-typedness. With attribute grammars, context-free grammars are combined with a different formalism to permit the description of such properties.

Definition (Context-free grammar)**.** A *context-free grammar* is a tuple (V,N,S,P) where V is a set of terminal symbols (the *alphabet*), N is a set of nonterminal symbols, S is the start symbol with $S \in N$, and P is a set of productions (defined below). The set V and N must be disjoint.

Definition (Production)**.** A (context-free) *production* $p = n \rightarrow \overline{m}$ is a rewrite rule with nonterminal symbol $n \in N$, and a sequence of symbols $\overline{m} \in V \cup N$. The sequence $\overline{m}$ may be empty. The application of p to a sequence of symbols $\overline{s} \in V \cup N$ constitutes to the rewriting of one occurrence of n in $\overline{s}$ to $\overline{m}$. In production p, n forms the left-hand side of the production and $\overline{m}$ the right-hand side.

To summarize, a grammar is a rewriting system where productions specify a rewrite step from a sequence of symbols to sequence of symbols. In a context-free grammar, a production specifies how to rewrite a single nonterminal symbol to a sequence of symbols. The rewriting terminates when only terminal symbols are left: when successive applications of productions to some singleton sequence n (with $n \in N$) results in a sequence of symbols, then this sequence is derived from n.

Definition (Sentence)**.** A *string* is a sequence of symbols. A *sentential form* is a string derivable from the start symbol of the grammar. A *sentence* is a sentential form consisting only of terminal symbols.

Definition (Derivation tree)**.** A derivation tree is a tree t that represents how a sentence $\overline{s}$ is derived from a symbol m, and is inductively defined as follows:

- A leaf t represents either the derivation of the empty string from a nonterminal symbol n if there exists a production $p = n \rightarrow \varepsilon$, or the trivial derivation of the singleton string $\overline{s} = v$ from a terminal symbol v. (Only) in the former case, we say that the leaf is associated with the nonterminal n and the production p. In the latter case, the leaf is only associated with the terminal v.

- If trees $t_1, ..., t_k$ represent the respective derivations of sentences $\overline{s_1}, ..., \overline{s_k}$ from symbols $m_1, ..., m_k$ then the tree t, formed by taking $t_1, ..., t_k$ as the respective children of the root, represents the derivation of the sentence $\overline{s_1} ... \overline{s_k}$ from symbol n if there exists a production $p = n \rightarrow m_1 ... m_k$. We say that the root of t is associated with the nonterminal n and the production p.

Definition (Syntax tree)**.** A *syntax tree* (or *parse tree*) is a derivation tree that is associated with the nonterminal n. This definition purposefully excludes singleton trees denoting a terminal symbol.

Definition (Abstract syntax tree)**.** An abstract syntax tree is the result of applying a projection to some syntax tree. Usually, the resulting tree has less branches (e.g. due to omission of layout) or branches replaced with a more general representation (e.g. desugared).

In this thesis, we do not concern ourselves with parsing, which is the process of constructing a syntax tree that represents the derivation of a given sentence. Instead, we assume the syntax tree as a given. In fact, we adopt the common convention to abstract over details in the syntax tree (e.g. whitespace) and work with abstract syntax trees instead.

$$\begin{array}{lll}
g ::= \textbf{grammar}\, N\, \overline{p} & & \text{-- the grammar for nonterminal } N \\
p ::= \textbf{prod}\, P\, \overline{s} & & \text{-- production } P\text{, with as RHS the sequence of symbols } \overline{s} \\
s ::= \textbf{term}\, x :: \tau & & \text{-- a terminal symbol with name } x \text{ and type } \tau \\
\quad \mid\ \textbf{nonterm}\, x : N & & \text{-- a nonterminal symbol with name } x \text{ and nonterminal } N \\
N, P, x & & \text{-- names of nonterminals, productions, and symbols}
\end{array}$$

Figure 1.11: Language to describe context-free grammars.

Definition (Language)**.** The *language* L_G specified by a grammar G is the set of sentences that can be derived from the start symbol of G.

If L_G is a programming language, then the source code of a program written in L_G is a sentence in L. Moreover, if L_G is a language of algebraic data types, then nonterminals describe type constructors, terminals describe primitive types, and productions describe data constructors. An AST represents a data structure, and the bit sequence in memory can be regarded as the sentence.

Context-free grammar notation. In Figure 1.11 we introduce a language for the *description* of context-free grammars: i.e. terms in this language can be interpreted as a grammer as defined above. We later extend the language to describe attribute grammars and some AG extensions. Figure 1.12 shows an example. We explain some aspects of the notation below.

Definition (Meta grammar)**.** When we talk about a grammar for a grammar, we call the former a *meta grammar*.

In the notation, a grammar is the composition of grammars for individual nonterminals. The set of terminals V and nonterminals N are left implicit, and productions P are grouped per nonterminal. We reuse these letters for other purposes, such as N as an identifier for a nonterminal, and P as an identifier for a production.

The grammar is abstract: instead of terminal symbols, only the *type* of a terminal symbol is given. To stress the difference between terminals and nonterminals, we use a double colon to specify the type of a terminal and a single colon to specify the name of a nonterminal.

Definition (Children)**.** Each symbol in the right-hand side of a production has an explicit name, which will be useful later. We call such named symbols the *children* of the production, which stresses the correspondence to children in the AST of nodes to which the production is associated.

In the notation, the nonterminal symbol of the left-hand side of a production is implicit, since we only describe the right-hand sides of productions. We give the symbol on the left-hand side of a production the fixed name *lhs*.

```
grammar String                    -- a cons-list of characters
  prod Nil                        -- empty list
  prod Cons term     hd :: Char   -- head of the list (hd is a terminal)
            nonterm tl : String   -- remainder (tl is a nonterminal)
```

Figure 1.12: Example of a context-free grammar.

Attribute grammars. We now extend the above formalism to denote context-free grammars with notation for attributes and their associated functions.

Definition (Attribute grammar)**.** An attribute grammar is a tuple (T,N,S,A,I,O,P,F), where the set of terminals T, set of nonterminals N, and start symbol S are defined as for a context-free grammar. The set A consists of attribute names. The map I associates a set of names $I_n \subseteq A$ with each nonterminal n in N, which make up the *inherited* attributes of n. Similarly, the map O associates a set $O_n \subseteq A$ with each nonterminal n in N, which makes up the *synthesized* attributes of n. For each n, the sets I_n and O_n must be disjoined. The productions $p \in P$ are redefined below. The set F consists of computable functions f which we call *semantic functions*.

Definition (Production)**.** An (attribute-grammar) production $p = u \rightarrow \overline{w} \cdot \overline{r} \cdot X$ consists of an annotated nonterminal symbol u and annotated symbols $\overline{w}$, rules $\overline{r}$, and a set of symbol names X.

Definition (Annotated symbol)**.** An *annotated symbol* is either an annotated terminal or nonterminal symbol. An *annotated nonterminal symbol* $u = x : n.\overline{a}$ is a combination of a distinct symbol name $x \in X$, a nonterminal symbol $n \in N$, and a collection of attribute names $\overline{a} \in A$ so that a is either in I_n or O_n. We say that n is associated to x. An *annotated terminal symbol* $x : v$ is a combination of a distinct symbol name $x \in X$ and a symbol $v \in V$.

Definition (Attribute occurrence)**.** A *reference to an attribute* $x.a$ is a combination of a symbol name $x \in X$ (associated to some nonterminal n) with an attribute name $a \in A$ so that either $a \in I_n$ or $a \in O_n$. A *reference to a terminal* x is a symbol name $x \in X$ which is associated to some terminal v. An *attribute occurrence* o is either a reference to an attribute or a reference to a terminal.

We call an occurrence $x.a$ also an attribute a of x. Attribute occurrences can be found in rules, which are defined below.

Definition (Rule)**.** A rule $\overline{o_1} = f\ \overline{o_2}$ of some production $u \rightarrow \overline{w} \cdot \overline{r} \cdot X$ consists of a semantic function $f \in F$ and attribute occurrences $\overline{o_1}$ and $\overline{o_2}$.

The occurrences $\overline{o_1}$ represent the attributes defined by the rule, which are synthesized attributes of u or inherited attributes of the children $\overline{w}$. The occurrences $\overline{o_2}$ represent the attributes used by the rule, which are the inherited attributes of u or synthesized attributes of the children $\overline{w}$. In addition, occurrences in $\overline{o_2}$ may also refer to terminals.

Due to the restrictions on occurrences it is always clear whether an occurrence references an inherited or a synthesized attribute. In our notation for attribute grammars (further below) we allow the same name to be used for an inherited and a synthesized attribute.

Decorated trees. To give a semantics to rules, we consider syntax trees annotated with attributes, and define how attributes of the tree are related to attribute occurrences, which are mentioned in the rules of productions.

Definition (Attributes of syntax trees)**.** An *annotated syntax tree* or *semantic tree* is a syntax tree T where in addition subtrees are associated with the smallest set Q defined as follows. Let t be a subtree and n be the nonterminal that is associated to t. For each attribute a in $I_n \cup O_n$, let there be a distinct attribute symbol $q_t \in Q$. The set Q represents the attributes of T.

The symbols q can be seen as instances of the attributes, or as occurrences of the attributes in the tree. This definition states that many trees may be associated with the same nonterminal yet have different instances of the attributes. Moreover, if t is an annotated syntax tree, then the attributes of each annotated direct subtree of t are a distinct subset of the attributes of t.

Definition (Attribute association)**.** Given some annotated syntax tree t with associated production p, an *attribute association* α is a mapping so that for each attribute occurrence o of p, either $\alpha\ o = q$ for some $q \in Q$ when o is a reference to an attribute (either of t or of a direct subtree of t), or $\alpha\ o = v$ when o is a reference to a terminal child v of t.

For each (node of an) annotated syntax tree, there exists such an attribute association. We leave open how this straightforward connection between attribute occurrences and attributes of the syntax tree is constructed.

Definition (Valuation)**.** A *valuation* M is a mapping that associates with each $q \in Q$ and each $v \in V$ a value in the host language, which is denoted as $M\ q$ or $M\ v$. Furthermore, $M\ v = [\![\ v\ ]\!]$ for $v \in V$ where $[\![\ v\ ]\!]$ is some encoding of v as a value in the host language. These values are called *decorations*.

Definition (Attributed syntax tree)**.** An *attributed* (or *decorated*) syntax tree is an annotated syntax tree combined with a valuation M.

A rule $\overline{o_1} = f\ \overline{o_2}$ encodes the condition $\overline{M\ (\alpha\ o_1)} = f\ \overline{M\ (\alpha\ o_2)}$ for each node the rule is associated with. Alternatively, we may say that f functionally defines occurrences o_1 in terms of occurrences o_2, and thus that inherited attributes of children are defined by the parent, whereas synthesized attributes of the children are defined by the children and may be used by the parent.

Definition (Correctly attributed syntax tree)**.** A syntax tree is *correctly attributed* when the conditions imposed by the rules are satisfied.

$I ::= \textbf{attr}\ N\ \bar{i}\ \bar{s}$	-- attribute declaration for nonterminal N
$i ::= \textbf{inh}\ y :: \tau$	-- declares inherited attribute y of type τ
$s ::= \textbf{syn}\ y :: \tau$	-- declares synthesized attribute y of type τ
$S ::= \textbf{sem}\ N\ \bar{c}$	-- semantics definition for N
$c ::= \textbf{prod}\ P\ \bar{r}$	-- semantics in the form of rules for prod P
$r ::= g\ [\overline{o_1}] = f\ [\overline{o_2}]$	-- a rule with pattern g and function f
$o ::= t.x.y$	-- attribute occurrence of child x and attribute y, and kind t
$t ::= inh \mid syn \mid loc$	-- explicit attribute kind (usually left implicit)
x, y	-- names of symbols and attributes
f, g	-- expressions

Figure 1.13: Minimalistic language for AGs.

attr *String*	-- declares atributes of *String*
inh *down* :: *Char*	-- an inherited attribute *down*
syn *sh* :: *String*	-- a synthesized attribute *sh*
sem *String*	-- rules for each production
prod *Nil* *lhs.sh* = *Nil*	-- rule 1: def. of syn attr of LHS
prod *Cons* *tl.down* = *loc.hd*	-- rule 2: def. of inh *tl* with term *hd*
lhs.sh = *Cons lhs.down tl.sh*	-- rule 3: of syn attr of LHS

Figure 1.14: Example of an AG that shifts a character in a string.

Notation for attribute grammars. The above definitions introduce concepts that underly attribute grammar languages and implementations. Figure 1.13 gives a minimalistic language for the description of AGs, which is an extension of the language for context-free grammars in Figure 1.11. We explain some of its aspects below.

The notation[5] in Figure 1.13 consists of a collection of nonterminal declarations g, attribute declarations I, and semantics blocks S. Attributes are declared separately for each nonterminal and have a type associated with them. The right-hand side of a rule is an *expression* $f\ [\bar{o}]$ in some formal language H with embedded references to attributes at identifier positions of H via attribute occurrences $\bar{o}$. The left-hand side of a rule is also an expression, but limited to patterns such as tuples. The use of expressions is slightly more flexible than just a function symbol.

For attribute occurrences $t.x.y$, we take the following notational conventions. The attribute kind t distinguishes inherited and synthesized attributes. This kind is always clear from the

[5] Note that we reused some letters here which we used before in a different context as the map I and the start symbols S.

context thus we usually leave it unspecified in examples, unless we want to stress the difference. To refer to an inherited attribute y of a symbol named x, we use $inh.x.y$ or simply $x.y$. To refer to a synthesized attribute y of a symbol named x, we use $syn.x.y$ or simply $x.y$. To refer to a terminal named x we use the attribute occurrence $loc.x.\mathbf{self}$ or simply $x.\mathbf{self}$ (see also Section 1.3.6).

Informally, an AG in this notation is well-formed when the description can be translated to an AG, and that the types of various identifiers and expressions are correct.

Figure 1.14 shows an example where we define a transformation on strings where each character is shifted one position to the right. We use an inherited attribute *down* to represent the preceding character. Given a value `'d'` as initial *down* value, the result for `"Ag"` is `"dA"`. We show a more complex example in Section 1.3.11.

Local attributes. We also use the notation $loc.loc.x$ to refer to a *local attribute* with the name x, which is an attribute defined by a rule of a production and is only in scope of that production. The name must be distinct from the name of a terminal symbol. Local attributes are typically used to represent common subexpressions.

Syntactic sugar. A terminal v can be encoded as a fresh nonterminal (Section 1.3.7) with a single synthesized attribute and a single ε-production that defines the attribute with the value $[\![\, v \,]\!]$. A local attribute can be encoded as a fresh nonterminal with a single inherited attribute and single synthesized attribute and a single ε-production that contains a rule that copies the value of the inherited attribute to the synthesized attribute. As convenient simplification, we therefore assume in the remainder of this chapter that terminal-leafs and local attributes are not present in AGs and ASTs, although we use terminals and local attributes in examples. In later chapters we take local attributes and terminals into account explicitly.

Interfacing with the AG. The evaluation of a tree described by the AG (which we discuss in Section 1.3.4) takes as input a record of values for the inherited attributes and results in a record with values of synthesized attributes.

1.3.2 Dependency Graphs

To describe how the values of attributes are actually computed, we consider the data dependencies induced by rules. The trees that we consider in this section are derivation trees generated by some AG (T,N,S,A,I,O,P,F).

Graphs of a production. A rule $\overline{o_1} = f\ \overline{o_2}$ represents a data dependency of occurrences $\overline{o_1}$ on occurrences $\overline{o_2}$, or equivalently, a flow of data from $\overline{o_2}$ via f into $\overline{o_1}$. These dependencies form a graph.

Definition (Production dependency graph)**.** A *Production Dependency Graph* (PDG) is a directed graph (V,E) associated with some production p. There is a one-on-one mapping between vertices $d \in V$ (Figure 1.15), and the nonterminal children of p, the rules of p and attribute occurrences in rules of p. The edges E consists of:

$d ::= o$		-- attribute occurrence vertex
$\mid$	**rule** x	-- rule vertex with some distinct identifier x
$\mid$	**child** x	-- child vertex with x the name of the child
x		-- identifiers

Figure 1.15: Syntax of vertices in a PDG.

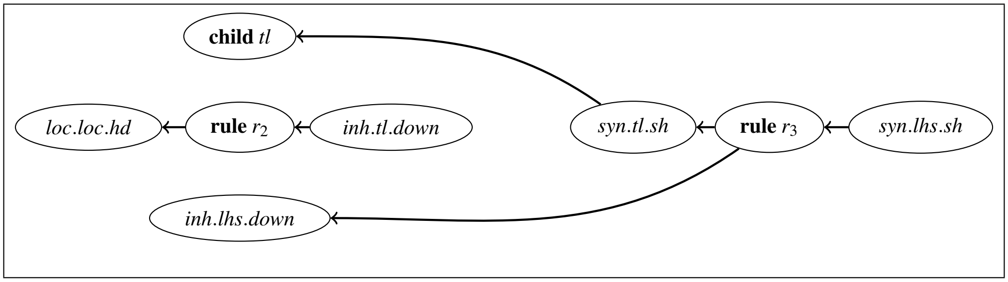


Figure 1.16: Exemplary PDG of production *Cons*.

- For each vertex *syn.x.y* an edge to a vertex **child** x.
- For each rule $[\overline{o_1}] = f\ [\overline{o_2}]$ (represented as vertex **rule** r) an edge from **rule** r to vertex o for each $o \in \overline{o_2}$, and an edge from vertex o to **rule** r for each $o \in \overline{o_1}$.

Similarly, a *production data-flow graph* is a production dependency graph with the edges reversed.

Figure 1.16 shows the PDG of the production *Cons* of Figure 1.14. For simplicity, we modelled the terminal *hd* as a local attribute *loc.loc.hd*.

Graph of a tree. These graphs can be projected on each node of a tree and then combined to form a dependency graph for a tree, or a data-flow graph for a tree. The general idea is that we take the PDG of the root of the tree and then add the graph for each child of the root, which describe the dependencies of synthesized attributes of the child on inherited attributes of the child.

Definition (Tree dependency graph)**.** For some annotated syntax tree t with associated production p, attribute association α, and annotated subtrees $t_1, ..., t_k$ with corresponding nonterminal children $c_1, ..., c_k$ of p, the *tree dependency graph* is inductively defined as the union of the PDGs of $t_1, ..., t_k$ and the instantiation of the PDG of p by transforming (with preservation of edges) rule and child vertices to fresh vertices and each occurrence vertex o to vertex $\alpha\ o$. Similarly, a *tree data-flow graph* is a tree dependency graph with the edges reversed.

Given a tree, evaluation algorithms of AGs (Section 1.3.4) are traversals over the tree dependency graph expressible as tree traversals[6] over the tree that are described by a non-deterministic tree-walking automaton (Section 1.3.3). Traditionally, such traversals may be demand-driven (traversing the dependency graph based on which attributes are needed) or may be described by a deterministic tree-walking automaton.

We come back to evaluation algorithms in Section 1.3.4. We first consider static approximations of the dependency graphs which can be used to prove that none of the attributes have a cyclic definition, and are an essential ingredient for a description of the evaluation with a *deterministic* tree-walking automaton.

Approximations. Below, we consider abstract interpretations of AGs that construct dependency graphs that are a static approximation of the tree dependency graphs of collections of trees in certain contexts:

Definition (Context)**.** A *context* is a symbol C of some fixed set of symbols Ω given per application and grammar. A context represents an additional set of invariants imposed on a tree.

Concretely, the invariants represented by a context may include that:

- The tree is associated with a certain nonterminal or production;
- The tree occurs as a subtree at certain position of a parent;
- The tree has attributes that are used according to some protocol [Farrow, 1984].

A collection of trees in a context share a common structure. By distinguishing contexts, we may consider projections of the tree dependency graphs of a collection of trees on the common attributes (of the root) that each tree in the collection has, so that the edges are superset of the projected edges of each individual tree. The projection-operation distributes over graph union, which ensures that we can work with approximations of projections of graphs of subtrees.

Definition (I/O graph)**.** An *I/O graph* of some nonterminal $n \in N$ is directed graph where the vertices consist of the attributes $a \in I_n \cup O_n$ and the edges represent either (indirect) dependencies or data flow between attributes of some trees that have n as root. An *I/O dependency graph* is an *I/O graph* where the edges represent data dependencies, and in an *I/O flow graph* the edges represent data flow.

Note that the above definition does not specify which edges are included in an I/O graph, but only specifies what the edges represent. We later give a consistency condition that specifies which edges must minimally be present.

Definition (Nonterminal dependency graph)**.** The *Nonterminal Dependency Graph* (NDG) of nonterminal n is the I/O dependency graph of n that approximates the dependencies between attributes of any *tree* associated with n.

[6] The shape of a tree dependency graph ensures that a description with a tree traversal is possible. The dependencies are properly nested: on a path from a synthesized attribute of a child to some (indirect) dependency, an inherited attribute of the child occurs before any attribute of a sibling or parent.

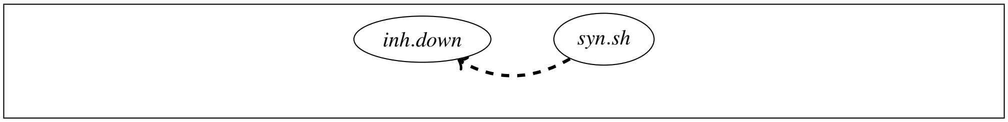


Figure 1.17: An I/O graph of nonterminal *String*.

An I/O graph of some nonterminal *n* serves as an approximation of the dependencies between attributes of *some* trees (context dependent) that are associated with *n*. In contrast, an NDG is a single, context-independent approximation of dependencies between attributes of some nonterminal.

Definition (Augmented PDG)**.** The *augmented PDG* of a PDG *G* of some production *p* and I/O graphs of the nonterminal children of the production, is *G* with edges added between attributes of children so that if there exists an edge between attributes of the given I/O graph of some child, then this edge also exists between the attributes of the child in the augmented PDG, and vice versa[7].

An augmented PDG of some production *p* serves as an approximation of the dependences between attributes of *some* trees that are associated with *p*. It is a PDG parameterized with the dependencies induced by the subtrees.

Collections of graphs. To specify which edges are part of the I/O graphs and augmented PDGs we consider some properties of collections of these graphs.

Definition (Consistent approximations)**.** A collection of I/O graphs and augmented PDGs is *consistent* when each I/O graph of some nonterminal *n* in some context $\mathcal{C}$ is a projection of the union of the augmented PDGs in context $\mathcal{C}$ of the productions associated with *n*.

Definition (Complete approximations)**.** A collection of I/O graphs and augmented PDGs is *complete* when for each production *p* the collection includes an augmented PDG and corresponding I/O graphs for each combination of contexts that the children of *p* can occur in.

Definition (Smaller approximations)**.** Graph *A* is *smaller* than graph *B* if A^* is a subgraph of B^*, where A^* and B^* are the respective transitively closed graphs of *A* and *B*.

Figure 1.17 gives an example of an I/O graph that is part of some collection that satisfies the above properties based on the AG in Figure 1.14. It is an I/O graph of the nonterminal *String* in the context of being the child *tl* of production *Cons*. The dependency from *syn.sh* on *inh.down* is induced by the production *Cons*. If Figure 1.16 would include an edge from *syn.tl.sh* to *inh.tl.down*, it would be an augmented PDG parameterized with the I/O graph.

[7]We specify here that the dependencies between attributes of a child in an augmented PDG match exactly to the dependencies between the attributes of the corresponding I/O graph that the augmented PDG is parameterized with, thus that edges of the PDG may impose constraints on the I/O graphs of the children.

Cycle analysis. Cycle analysis of AGs is an abstract interpretation [Nielson et al., 1999] that approximates the tree dependency graph of any tree by constructing a complete and consistent collection of I/O graphs and augmented PDGs. The consistency and completeness requirements lead to a set of mutually recursive equations for which we want to compute a least solution. Such a solution is obtained with fixpoint iteration. In Chapter 4 we consider a concrete cycle analysis; here we look only at the general structure of such analyses.

Various approaches distinguish different contexts which influence the accuracy and complexity of the approximations:

Uniform AG. Knuth [1968] distinguishes as contexts the nonterminals to which trees are associated, and the different positions in the right-hand sides of productions where nonterminals can occur in. This approach[8] leads to one I/O graph per nonterminal and one augmented PDG per production. The AG is *uniform* or *absolutely non-circular* if the augmented PDGs in this flavor are acyclic.

Well-defined AG. Knuth [1971] additionally distinguishes the production to which the tree is associated. This approach leads to one I/O graph per nonterminal and production, and an exponential number of augmented PDG, one for each combination of children with productions. An AG is *well-defined* if for any tree the tree dependency graph is cycle-free, which is the case when the augmented PDGs in this flavor are acyclic.

Ordered AG. Kastens [1980] distinguishes as contexts only the nonterminals to which trees are associated, but does not differentiate the occurrences of nonterminals in the right-hand sides of productions.

In our experience with UUAG and in agreement with observations by Räihä and Saarinen [1982], an AG is in practice also ordered when it is well-defined.

1.3.3 Tree-Walking Automata

Tree-walking automata arose from tree language theory and were introduced by Aho and Ullman [1969]. A tree-walking automaton (TWA) is device that walks over a tree in a contiguous manner and is accompanied by a state machine that describes how the nodes of the tree change their state upon each visit and whether the device goes up to the parent or goes down to one of the children as the next step. Section 1.3.4 uses TWAs to describe evaluation algorithms of AGs. We keep here a simplistic presentation; there exist many extensions that increase the expressiveness of these automata, such as pebbles [Engelfriet and Hoogeboom, 1999].

Definition (Tree-walking automaton)**.** A TWA for some AST is a tuple (V, Q, I, F, δ), with an alphabet V of node labels, a finite set of states Q, an initial state $I \in Q$, a set of final states F, and a transition relation $\delta \subseteq (V \times Q \times Q \times C \times Q)$, where $C = \{up, down_0, ..., down_k\}$ is a set of commands and k is the maximum branching factor of nodes in the AST. There exists a

[8] The actual approach does not distinguish different positions but instead instantiates augmented PDGs with I/O graphs (thus copies them).

$p ::= \textbf{plans}\; P : N\; \overline{v}$	-- execution plan for production P of nonterminal N
$v ::= \textbf{visit}\; C\; \overline{i}\; \overline{s}\; \overline{b}$	-- a visit in context C that may need $\overline{i}$ and can produce $\overline{s}$
$b ::= r$	-- evaluation rule
$\quad\mid\; \textbf{invoke}\; x\; C\; \overline{i}\; \overline{s}$	-- a *down* x in context C, which can provide $\overline{i}$ to x and expects $\overline{s}$
x	-- child name
C	-- context identifier
i, s, r	-- as defined in Figure 1.13

Figure 1.18: The execution plan language.

one-to-one relation between productions and symbols in V so that each node is labelled with a $v \in V$ depending on the production associated with the node.

A tuple $(v, q, q_0, c, q') \in \delta$ represents a transition from a node labelled v in state q to a state q', and moving to the node according to c. The automaton keeps track of a bit of history: q_0 is the state of the node that caused the transition to the current node.

Definition (Deterministic tree-walking automaton)**.** A *deterministic TWA* is a TWA where the transition relation is a *function* $\delta :: V \to Q \to Q \to (C, Q)$. Otherwise the TWA is *nondeterministic*.

Acceptance. Initially each node in the AST is associated with the initial state I. The automaton starts at the root of the tree and stops if no step can be taken anymore. The tree is accepted if the automaton ends with the root having an associated state in F. With each step, the automaton visits a node. If the automaton is at a node with label v and associated state q, and previously visited a node in state q_0, then the automaton chooses a step c and new state q' so that $(v, q, q_0, c, q') \in \delta$, or the automaton stops if no such step exists. In the former case, the automaton updates the state of the node to q' and visits the parent if $c = up$ or visits child i if $c = down_i$.

Evaluation of rules. An actual AG evaluation algorithm does not only traverse the tree but also needs to apply rules to compute attributes. Thus, in an actual implementation, the automaton also applies a subset $\gamma\,(v, q)$ of the production's rules upon making a transition to state q at a node with label v.

Definition. Visit The TWA *visits* a node n if it arrives at n and executes $\gamma\,(v, q)$ where v is the label of n and q is the state of n.

Implementation with the Zipper. Various forms of deterministic TWAs can be implemented in a purely functional programming language using the *zippers* [Huet, 1997]. Such an approach models the imperative updates of the automaton to the state.

```
plans Nil : String
   visit   AnyCtx   inh down :: Char   syn sh :: String
      lhs.sh   = Nil
plans Cons : String
   visit   AnyCtx   inh down :: Char   syn sh :: String
      tl.down = loc.hd
      invoke tl AnyCtx   inh down :: Char   syn sh :: String
      lhs.sh   = Cons lhs.down tl.sh
```

Figure 1.19: Exemplary execution plans of nonterminal *String*.

Execution plans. Figure 1.18 introduces a language of execution plans $\overline{p}$ for the description of the transition relation of TWAs, of which we explain some aspects below. Figure 1.19 shows an example. The language is not expressive enough to describe all transition relations, but it suffices for a description of an AGs evaluation.

A collection of plans-blocks represents the transition *relation* δ. Since a context is an agreement between parent and child, a context models the q_0 parameter of δ. A plans-block is associated to a unique production P and consists of a number of visit-blocks. A visit-block v describes visits to the node in context C, and thus represents a subset $\delta^v_{P_C}$ of δ_{P_C}. Let $\overline{r}$ be the rules that can be evaluated as a consequence of the tree walk taking transitions from δ^v_P. Then the inherited attributes $\overline{i}$ of the current node may be needed in the evaluation of $\overline{r}$ and the synthesized attributes $\overline{s}$ can be computed by $\overline{r}$. An invoke-rule represents possible transitions to some child x of P in some context C so that values of inherited attributes $\overline{i}$ of x can be provided, and values of synthesized attributes $\overline{s}$ of x may be needed by the current node.

Given an acyclic PDG of P, the relation δ_P and the accompanying subset of γ can be generated. This procedure is left as an exercise to the reader; we note that the number of states is possibly exponential in the size of the productions and their children, and the order of appearance of rules is irrelevant for the translation.

A collection of plain-blocks may represent also a transition *function* δ. In this case, each visit-block v represents a distinct $\delta^v_{P_C} \subseteq \delta_{P_C}$ with precisely $1+n$ elements (of which element $i+1$ can be thought of as the continuation after visiting child number i), where n is the number of invoke-rules in a visit-block. The rules must occur in define-before-use order. The example in Figure 1.19 satisfies these constraints.

Implementations of execution plans. In comparison to the Zipper, there are less 'imperative' and more efficient encodings of TWAs in functional languages. If we consider TWAs with transition relations that do not make use of the q_0 parameter, and thus do not distinguish contexts, Swierstra and Alcocer [1998] presented an approach by exploiting lazyness (for nondeterministic TWAs) which we sketch in Section 1.3.4, and Saraiva and Swierstra [1999] presented visit functions, which are coroutines encoded as continuations (for deterministic

TWAs with a total order imposed on visits which we sketch in Section 1.3.5).

Implementations described in this thesis. In Chapter 3 we build upon visit functions. In Chapter 4 we show that we can represent transition relations that use contexts. In later chapters we look at extensions to AGs that have accompanying evaluation algorithms that cannot be described with a deterministic TWA. For example, in Chapter 5 we allow tree walks directed by values of attributes, and in Chapter 7 we allow tree walks to jump back and forth saved positions.

1.3.4 Demand-driven Attribute Evaluation

We can express demand-driven attribute evaluation by mapping an AG onto an execution plan that describes a nondeterministic TWA (Section 1.3.3). This translation is straightforward:

- From each production an execution plan is derived with a single visit that lists all attributes of the productions left-hand side.
- The visit-block contains the rules of the production, and an invoke-rule per child which lists all the attributes of the child.

The attributes are then computed by running the described TWA.

An execution plan in the above form has a straightforward translation to algebras Haskell. We sketch this translation. It is optional background material, but is not required for the understanding of later chapters.

Catamorphisms. Let F be an endofunctor so that data constructors describing the AST A form the initial F-algebra. The execution plans can be mapped straightforwardly to an F-algebra ϕ so that *cata* ϕ A is a function g (which we call the *semantic result* or *semantic tree*) that takes an argument for each inherited attribute of the root of A, and provides a result for each synthesized attribute of the root. Function g encodes the tree dependency graph of *ast* (Section 1.3.2), and lazy evaluation acts as nondeterministic TWA, with the additional feature that each rule in the execution plan is at most executed once. Effectively, the functional program is a term-graph representation of the dependency graph, and evaluation, rewriting this term-graph [van Eekelen et al., 1996], results in the values of synthesized attributes.

Haskell translation. Swierstra and Alcocer [1998] showed how to express the algebra in Haskell as a function (called a *semantic function*[9]). We demonstrate this approach based on the example in Figure 1.14.

Figure 1.20 gives a sketch of the translation to Haskell code. We explain some aspects of this example below.

For each nonterminal N a type $T\_\langle N\rangle$ for the semantic result of a tree associated with N is generated, which is *T_String* in the example. Nonterminal *String* has an inherited attribute of

[9] Depending on the context, a semantic function may be the function in a productions rule or a function corresponding to some data structure in an algebra.

```
type T_String = Char → String                       -- type of the node's semantics
cata_String :: String → T_String                    -- maps AST to semantic AST
cata_String Nil           = sem_Nil
cata_String (Cons hd tl) = sem_Cons hd (cata_String tl)
sem_Nil :: T_String                                 -- production without children
sem_Nil lhs_down = (lhs_sh) where ...               -- note that the T-type is a function
sem_Cons :: Char → T_String → T_String              -- production with two children
sem_Cons loc_hd loc_tl lhs_down = (lhs_sh) where ...
```

Figure 1.20: Sketch of the algebra.

```
sem_Cons loc_hd loc_tl = λlhs_down →
  let lhs_down = (lhs_sh) where
     tl_down  = loc_hd                      -- transcription of the first rule
     (tl_sh)  = loc_tl tl_down              -- recursive call to child tl
     lhs_sh   = Cons lhs_down tl_sh         -- transcription of the second rule
  in (lhs_down)
```

Figure 1.21: The body of *sem_Cons*.

type *Char* and a synthesized attribute of type *String*, hence the type of *T_String* is *Char* → *String*.

The *cata*-function associates a semantics with each constructor of the AST. It replaces a constructor *P* with its semantic variant *sem_* $\langle P \rangle$.

Semantic functions. Figure 1.21 sketches the generated semantic function *sem_Cons*. The body of a semantic function encodes the productions rules and the invoke-rules as given in the execution plan. The encoding of the rules is straightforward. An invoke-rule with some child *tl* is translated to a recursive call to the parameter that represents the semantic tree of *tl*. The call is parameterized with values of the inherited attributes of *tl*, and a pattern match against the results extracts the values of the synthesized attribute of *tl*.

Remarks. Demand-driven evaluation of AGs is popular in current AG systems. An advantage of on-demand evaluation is that it does not require an abstract interpretation as part of its implementation, and works in combination with extensions such as remote reference attributes [Magnusson and Hedin, 2007].

Another advantage is that attributes are not computed when their values are not needed at runtime. Demand-driven evaluation may produce values of attributes even in the presence of cyclic attribute dependencies.

A disadvantage of demand-driven evaluation is the potential high space requirements also known as space leaks.

The approach based on lazy evaluation goes further than demand-driven attribute evaluation because the set of required attributes depends on how the values of attributes are inspected. In fact, using lazy lists it is possible to associate countable infinite attributes with a nonterminal.

1.3.5 Statically Ordered Attribute Evaluation

We can express a statically ordered evaluation of AGs by mapping the AG onto a deterministic TWA (Section 1.3.3). Depending on what contexts are distinguished and the strictness properties of the rules, there may be values computed for attributes that are not needed for the result, or which are only needed later. Therefore, we will call this also eager, greedy or strict evaluation of AGs.

Definition (Multi-visit AG)**.** A *multi-visit AG* is an AG for which a statically ordered evaluation strategy is possible.

Multi-visit AGs play an important role in this thesis because they make the notion of phasing explicit, which is useful for reasoning about what parts of the tree have been investigated and constructed so far.

Attribute scheduling. To map an AG to an execution plan of a deterministic TWA, we need to determine for each production P how trees are visited that are associated to P. For each visit, we need to determine which rules to apply and how the children of P are visited. This process requires acyclic augmented PDGs.

Scheduling algorithms. Kastens [1980] presented an approach that attempts to derive a smallest single sequence $Delta_N$ of visits per nonterminal N so that the attributes of a child with some nonterminal N can be computed by visiting the child according to some prefix of $Delta_N$. Effectively, this approach does not distinguish any contexts. The approach entails adding edges to the I/O graphs of N so they are equal and form a total order on the attributes, which is possible for most AGs in practice, but the approach of Kastens sometimes needs help in the form of additional attribute dependencies to accomplish this.

Kennedy and Warren [1976] presented an approach that works for any absolutely noncircular AG. Their approach distinguishes protocols as contexts, which are the possible orders in which the parent provides inherited attributes and demands synthesized attributes. In Chapter 4 we investigate this approach in-depth and present a translation to Haskell.

Coroutines. A deterministic TWA can be implemented with coroutines [Warren, 1976].

Definition (Coroutine)**.** A *coroutine* [Marlin, 1980] is a function that can pause during its execution and return results to the caller. It may be parameterized with additional arguments when resumed by the caller. A *generator* is a coroutine that does not take additional arguments.

When no contexts are distinguished, such coroutines can be encoded in a purely functional language as *visit functions* [Saraiva and Swierstra, 1999]. The body of such a function builds a continuation that is used for a subsequent call and returns it as part of the result. From the caller's perspective:

```
...
$f\_visit_1 = s\_f$    -- assuming $s\_f$ is the coroutine of a child $f$
$(f\_syn_1, f\_syn_2, f\_visit_2) = f\_visit_1\ f\_inh_1\ f\_inh_2$
$(f\_syn_3, f\_syn_4, f\_visit_3) = f\_visit_2\ f\_inh_2\ f\_inh_2$
...
```

The callee has the following structure where the dots represent the usual encoding of the rules that are scheduled to a particular visit:

```
$sem\_\langle P\rangle\_1\ s\_child_1\ s\_child_2 = lhs\_visit_1$ where
  $lhs\_visit_1\ inh_1 \ldots inh_2 = (syn_1, \ldots, syn_2, lhs\_visit_2)$ where
    ...
    $lhs\_visit_2\ inh_3 \ldots inh_4 = (syn_3, \ldots, syn_4, lhs\_visit_3)$ where
      ...
```

We explain this encoding in great detail throughout this thesis. Many chapters of this thesis present variants of this encoding.

Remarks. Historically, statically ordered attribute evaluation results in faster code and less memory usage. Also, recent developments on multi-core computing may give renewed interest in visit sequences with respect to parallel evaluation [Wang and Ye, 1991].

1.3.6 Incremental Descriptions

Our formalism allows us to write various declarations that together form an AG in any order. This is a consequence of the purely functional relation between attributes. It is an important property of AGs, because it allows us to incrementally and separately describe AGs. In this thesis, we make repeatedly use of this feature to eliminate common patterns from examples. The separate descriptions are simply be merged by string concatenation.

Incremental notation. An AG description is incremental: nonterminals, productions, children, attributes and rules may be declared separately. At the same time, we may declare productions and attributes for multiple nonterminals, and children and rules for multiple productions.

Figure 1.22 gives a number of examples. A declaration of a nonterminal and production may appear multiple times and provide additional declarations. Nonterminal *Expr* has only one production *App* but its contents are determined by several declarations.

```
grammar Expr                                -- productions for Expr
   prod App       term impred :: Bool      -- with terminal for App
   prod Var       term nm      :: Name      -- with terminal for App
   prod App Var   term uid     :: Int       -- extra terminal for App and Var
grammar Expr Type                           -- productions for multiple nonterminals
   prod App       nonterm f, a : self       -- multiple nonterminals of the same type
attr Expr         inh x, y :: Int           -- multiple attributes of the same type
attr Expr Type    syn output :: self        -- attributes for multiple nonterminals
sem Expr Type                               -- rules for multiple nonterminals
   prod App       lhs.output = f.output `mkApp` a.output
sem Expr
   prod App Var lhs.x = 1                   -- rule for multiple productions
```

Figure 1.22: Examples of incremental notation.

The type **self** is special and represents the type of the actual nonterminal of the description the **self** appears in. The function *mkApp* must thus be an overloaded function that works both on *Expr*s and on *Type*s.

The merging process is straightforward[10]. A duplicate declaration of an attribute of the same nonterminal is not allowed and considered a static error. Similarly, after merging, attributes may not be defined by more than one rule, and each child of a production must be defined once.

Nonterminal sets. To aid the definition of attributes on many nonterminals, we may use nonterminal sets. We use nonterminal sets often in actual code but only sporadically in this thesis.

Definition (Nonterminal set)**.** A *nonterminal set* is a nonterminal name that represents one or more other nonterminals or sets.

For example, we define a nonterminal name *AllExpr*, which actually stands for *Expr* and *Decl*. When we declare attributes on *AllExpr*, these are actually declared for *Expr* and *Decl*:

set *AllExpr* : *Expr Decl* -- *AllExpr* includes $Expr \cup Decl$.

Nonterminal sets are extensible: a set declaration of some set *N* may appear multiple times in an AG description. Additionally, notation for set union and set difference may be used to define sets. Determining sets is a straightforward fixpoint computation.

[10] The interested reader may take a look at `Transform.ag` in the `uuagc` project for a merge algorithm. This algorithm also allows some declarations to overwrite previous declarations.

Nonterminal inheritance. A nonterminal may masquerade as a set. If due to a set declaration, such a set includes other sets and nonterminals, the nonterminal inherits their attributes, productions, and rules.

1.3.7 Higher-Order Children and Attributes

Definition (Semantics of a child)**.** The *semantics of a child* with a nonterminal N is a semantic tree associated to nonterminal N. The decorations still have to be given.

Definition (Higher-order child)**.** A *higher-order child* is a child with a semantics determined by the value of an attribute.

A conventional child is determined by syntax, whereas a higher-order child is determined by an attribute. Higher-order children are also known as higher-order attributes or nonterminal attributes. The notion 'higher order' originates from being able to pass the semantics of children around as first class values.

Higher-Order AGs (HOAGs) [Vogt et al., 1989] support higher-order children. As part of this thesis, we implemented this feature in UUAG [Löh et al., 1998][11]. Higher-order children play an essential role in this thesis: with such children we can dynamically grow the tree (e.g. a proof tree) instead of being limited to a fixed tree (e.g. the parse tree).

Children defined by rules. In a conventional AG, the semantics of a child of a production is determined prior to attribute evaluation. In a Higher-Order AG (HOAG), additional higher-order children may be declared for a production. Their semantics is the value of an attribute, or alternatively, the outcome of evaluating a rule:

child $x : N = f\ [\overline{a}]$ -- rule that introduces a child x

The expression $f\ [\overline{a}]$ evaluates to the semantics for x as the following example demonstrates:

```
child x : String = sem_Nil   -- declares a child x that is defined by sem_Nil
x.down           = 'z'       -- inherited attr of x
```

Implementation. We see in later chapters how child-rules can be implemented. In the translation to Haskell as sketched in Section 1.3.4, the semantics of a child is a function from inherited to synthesized attributes, and each child is translated to a function call. A child-rule in this section is a conventional evaluation rule that defines some local attribute, where the local attribute is used as the function to call:

```
loc_x  = sem_Nil         -- local attr determines the semantics of x
x_down = 'z'             -- defines inherited attr of x
x_sh   = loc_x x_down    -- call to child x
```

[11] The syntax that we use here deviates slightly from the actual syntax of higher-order children in UUAG. A type signature **inst**.$x : N$ declares a child x, and a conventional rule must define the attribute **inst**.x with the semantics for x. In addition, when x already exists, its definition is a function that transforms the original semantics of x.

Desugaring. HOAGs can be used to desugar an AST. As example of an HOAG, suppose that *String* has a special production *Single* for single-character strings. Instead of defining the semantics directly for *Single*, we add a child *repl* that represents the string in terms of *Cons* and *Nil*:

```
grammar String prod Single    term x :: Char    -- the Single production
sem String prod Single                          -- and its semantics
   child repl : String = sem_Cons loc.x sem_Nil -- higher-order child
   repl.down           = lhs.down               -- inh attr of child repl
   lhs.sh              = repl.sh                -- syn attr of child repl
```

Multi-visit AGs as HOAGs. In a multi-visit AG, a child may be visited multiple times to compute some of the attributes. Such an AG can be encoded as a HOAG, which we show below. As we see in later chapters, we worked on a core language that can represent such AGs, and the question arose whether to use HOAGs as a target language. We did not do this because of other requirements, but we present the translation anyway since it may give some insight in how we organized the visit functions earlier.

We may encode multiple visits to some child c as a single visit to child c that only requires the inherited attributes of the first visit and only provides the synthesized attributes of the first visit. Additionally, it produces an attribute $c.cont$ that represents the semantics of c after the visit. For the second visit, we use a higher-order child c_2 with $c.cont$ as semantics. We then visit c_2 to provide/obtain the attributes of the second visit. For the next visit we use $c_2.cont$, etc. This approach requires the introduction of a potential large number of new nonterminals.

For a nonterminal N with m visits, we introduce the nonterminals $N_1, ..., N_m$, such that each nonterminal N_i has the inherited and synthesized attributes as associated to visit i of nonterminal N. In addition, N_i (for $i < m$), has an extra synthesized *continuation* attribute *cont* that contains the semantics of N_{i+1}.

For a production P we introduce the productions $P_1, ..., P_m$. The terminals of production P_i represent the decorations as available prior to visit i. Thus, P_1 consists of the original terminals and nonterminals of P, and P_i (for $i > 1$) consists of the terminals of P_{i-1} and additionally has terminals which encode the attributes computed in visit i. As optimization the terminals that are not needed in later visits can be omitted from P_i.

The semantics for P_i consists of the rules in P's plan for visit i (modulo renaming of attribute references). Additionally, a rule is added which computes the semantics of P_{i+1} and stores it in attribute $lhs.cont$.

In this translation, a visit to a child (for $i > 1$) is thus represented as a higher-order child that is instantiated by the continuation attribute produced by the previous visit.

1.3.8 Circular Reference Attributes

Reference Attributed AGs (RAGs) are an extension of attribute grammars with (remote) reference attributes [Magnusson and Hedin, 2007]. This is a common extension of AGs that utilize a demand-driven evaluation algorithm. The extension allows subtrees to be passed around in attributes. Normally, the attributes of such a subtree *T* can be used and defined only by the direct parent of *T*. However, with the extension, the attributes of *T* may be used and defined by any rule holding a reference to *T*. Such an attribute is said to be accessed *remotely*.

Graph structure. With this extension, the nodes are organized in a graph structure instead of a tree. Calculations over graphs often require some form of repetition, which can be encoded with cyclic attributes. Values of cyclic attributes can be computed if the demand-driven evaluation is extended with fixpoint iteration, and the attributes are given an initial value. The computation terminates if the rules are monotonic and each ascending chain of attribute values stabilizes.

Advantages and disadvantages. Reference attributes provide a convenient way of transporting information from one location in a tree to another location. Also, the extension allows more analyses to be modelled with AGs, such as abstract interpretations, which are typically fixpoint iterations over graphs. This expressive power comes with a price: the well-definedness of an AG cannot be statically verified in general (Section 1.3.2). Moreover, if values of *inherited* attributes can be defined remotely, well-formedness of the AG cannot be checked statically, which has as consequence that a straightforward mapping (such as in Section 1.3.4) to a purely functional language is not possible[12].

1.3.9 Correspondences between AGs, HOAGs, Monads, and Arrows

In later chapters, we translate AGs to a monadic target language [Meijer and Jeuring, 1995] and also consider AGs translated to Arrows [Hughes, 2004]. Being able to structure a computation as a monad or arrow allows reflection on the structure of the computation. We use such introspection in Chapter 7 to implement a step-wise evaluation strategy.

For example, consider the code of production *Cons* of the example in Section 1.3.1. In its present formulation, it can be evaluated in a strict fashion:

```
$sem\_Cons\ loc\_hd\ loc\_tl\ lhs\_down = (lhs\_sh)$ **where**
  $tl\_down = loc\_hd$
  $(tl\_sh)\ \ = loc\_tl\ tl\_down$
  $lhs\_sh\ \ = Cons\ lhs\_down\ tl\_sh$
```

We rewrite this code using arrow notation [Paterson, 2001] and call the result an execution plan:

```
$sem\_Cons\ field\_hd\ loc\_tl =$ **proc** $lhs\_down \rightarrow$ **do**
  $tl\_down \leftarrow f_{copy}\ \prec field\_hd$                -- transcription of the first rule
  $tl\_sh\ \ \ \ \leftarrow loc\_tl \prec tl\_down$             -- invoke arrow of child tl
  $lhs\_sh\ \ \leftarrow f_{cons}\ \prec (lhs\_down, tl\_sh)$   -- transcription of the second rule
  $returnA \prec lhs\_sh$                                      -- output
```

12 We implemented synthesized reference attributes in UUAG for lazily evaluated grammars.

$$\begin{aligned}\textbf{where}\, f_{copy} &= id\\ f_{cons} &= uncurry\ Cons\end{aligned}$$

The translation into arrow notation essentially desugars the above code into point-free style, thus in a linear composition of the rule functions that is interspersed with combinators to rearrange the intermediate attribute values.

Can we represent any AG in this way? When we limit the expressiveness of the language of the rules to tree constructions only, conventional AGs can express primitive recursive functions whereas HOAGs can express all computable functions. This difference can be observed when translating to arrows. In this setting, the semantics of a tree is an arrow that takes a tuple of inherited attributes as input and produces a tuple of synthesized attributes as output. A conventional AG is expressible as a plain arrow, but a HOAG requires the generalization to a monad, and AGs that are not statically ordered may require a feedback-loop.

Introspection of the arrow is possible using defunctionalization [Reynolds, 1998]. Such an arrow actually encodes the tree dependency graph (Section 1.3.2), thus introspection on this structure allows a runtime optimization (e.g. elimination of identity functions) of this graph which may be useful when traversing parts of the graph several times (Chapter 5).

1.3.10 Specification of Typing Relations

An AG can serve as a specification of a type system. We can straightforwardly map an AG to a set of type rules (inference rules). The other way around (Section 1.4), which is the concern of this thesis, is not as straightforward, because there may not exist an inference algorithm, and the type rules may be under-specified (Section 1.2).

We associate nonterminals with typing relations. In the mapping of an AG to type rules, we introduce a relation $I_N \vdash N : O_n$ for each nonterminal N, where I_N and O_n are the inherited and synthesized attributes of N respectively. The N represents the abstract syntax.

We associate productions with type rules. For each production **prod** $P\ \overline{s}$ we introduce a type rule. The judgments of the rules are derived from the children and the rules of P.

For each child **nonterm** $c : N$ in $\overline{s}$, we add the judgment $\overline{c_I} : c : \overline{c_S}$ to the type rule. As conclusion, we introduce the judgment $\overline{lhs_I} : lhs : \overline{lhs_S}$. This introduces all attributes and children, but requires the attributes to be constraint according to the rules, which we accomplish by mapping each rule straightforwardly to an equivalence judgment.

Finally, we need to specify how each child c is obtained from the AST lhs. The judgment $Con_P\ (lhs, \overline{c})$ caters for that.

1.3.11 Damas-Hindley-Milner Inference

As an example of a type inference algorithm written with attribute grammars, we give an AG implementation of the DHM algorithm (Section 1.2.6) by using the same approach as presented by Dijkstra and Swierstra [2004].

Definition (Chained attribute)**.** A *threaded attribute* or *chained attribute* stands for both an inherited and a synthesized attribute with the same name.

```
grammar Expr                          -- abstract grammar for expressions
  prod Var  term     x    :: Name     -- the identifier x
  prod App  nonterm  f    : Expr      -- left-hand side of application
            nonterm  a    : Expr      -- right-hand side of application
  prod Lam  term     x    :: Name     -- the identifier x bound by the lambda
            nonterm  b    : Expr      -- the body of the lambda
  prod Let  term     x    : Expr      -- the name of the binding
            nonterm  e,b  : Expr      -- the binding and body
attr Expr   inh env    :: Env         -- inherited environment
            chn subst  :: Subst       -- chained attr: both inh and syn
            syn ty     :: Ty          -- synthesized type
            syn errs   :: Errs        -- collection of type errors
```

Figure 1.23: DHM grammar and attributes.

The environment is modeled as an inherited attribute, errors as a synthesized attribute, and the substitution as a chained attribute.

For the type we have two options. Either the parent passes an expected type to the child that is further constrained by the child, or the child passes up an inferred type that is further constrained by the parent. When type annotations are part of the language, the former approach detects type errors faster. The information from these type annotations can then be given to a child, instead of verifying the resulting type after the fact. However, for this example it does not matter, thus we take the latter approach, which we also used for the monadic implementation of DHM in Section 1.2.6.

Figure 1.23 shows the grammar and attributes of the example. The chained attribute *subst* is shorthand for an inherited and synthesized with both *subst* as name. To make it clear which of the two attributes we intend, we explicitly prepend *syn* and *inh* to the name in the rules.

The rules in Figure 1.24 describe how the environment is passed top-down through the tree, how the substitution is threaded in-order through the tree, and how errors are collected bottom-up. Since functions such as *instantiate* and *unify* produce an updated substitution, it is the threading of the substitution that determines in what order the effects of these operations are visible in the substitution. The substitution needs to be threaded carefully in order not to loose any constraints on type variables.

In contrast to the monadic approach in Section 1.2.6, the rules are compositional. The distribution of the environment, the threading of the substitution, and the collection of error messages can be described separately and relatively independently. On the other hand, the rules are also more verbose because the environment, substitution and error messages are not hidden. Furthermore, rules that employ *generalize* are crosscutting as they deal with types, substitutions and environments. This has a negative effect on the degree of separation in the descriptions of these individual attributes. We address both issues in Section 1.3.12.

$$
\begin{array}{lll}
\multicolumn{3}{l}{\textbf{sem}\ \mathit{Expr}} \\
\quad\textbf{prod}\ \mathit{Var} & \mathit{loc.scheme} & = \mathit{lookup\ loc.x\ lhs.env} \\
 & (\mathit{lhs.ty}, \mathit{syn.lhs.subst}) & = \mathit{instantiate\ loc.scheme\ inh.lhs.subst} \\
 & \mathit{lhs.errs} & = \emptyset \\
\quad\textbf{prod}\ \mathit{App} & \mathit{a.env} & = \mathit{lhs.env} \\
 & \mathit{f.env} & = \mathit{lhs.env} \\
 & (\mathit{loc.res}, \mathit{inh.a.subst}) & = \mathit{fresh\ inh.lhs.subst} \\
 & \mathit{inh.f.subst} & = \mathit{syn.a.subst} \\
 & (\mathit{loc.errs}, \mathit{syn.lhs.subst}) & = \mathit{unify\ f.ty}\ (\mathit{a.ty} \rightarrow \mathit{loc.res})\ \mathit{syn.f.subst} \\
 & \mathit{lhs.errs} & = \mathit{f.errs} +\!\!+\ \mathit{a.errs} +\!\!+\ \mathit{loc.errs} \\
\quad\textbf{prod}\ \mathit{Lam} & \mathit{b.env} & = \mathit{insert\ loc.x\ loc.argty\ lhs.env} \\
 & (\mathit{loc.argty}, \mathit{inh.b.subst}) & = \mathit{fresh\ inh.lhs.subst} \\
 & \mathit{lhs.ty} & = \mathit{loc.argty} \rightarrow \mathit{b.ty} \\
 & \mathit{syn.lhs.subst} & = \mathit{syn.b.subst} \\
 & \mathit{lhs.errs} & = \mathit{b.errs} \\
\quad\textbf{prod}\ \mathit{Let} & \mathit{e.env} & = \mathit{lhs.env} \\
 & \mathit{b.env} & = \mathit{insert\ loc.x\ loc.scheme\ lhs.env} \\
 & \mathit{loc.scheme} & = \mathit{generalize\ lhs.env\ e.ty\ syn.e.subst} \\
 & \mathit{inh.e.subst} & = \mathit{inh.lhs.subst} \\
 & \mathit{inh.b.subst} & = \mathit{syn.e.subst} \\
 & \mathit{syn.lhs.subst} & = \mathit{syn.b.subst} \\
 & \mathit{lhs.errs} & = \mathit{e.errs} +\!\!+\ \mathit{b.errs}
\end{array}
$$

Figure 1.24: DHM with AG rules.

1.3.12 Copy Rules and Collection Attributes

We often pass values in standard top-down, bottom-up, and in-order patterns between attributes of the tree. The rules that encode these patterns are trivial: essentially identity-functions between attributes. For example, to pass an environment topdown to the children of an application we use the following rules:

```
sem Expr prod App
  inh.left.env  = id inh.lhs.env   -- copy down left
  inh.right.env = id inh.lhs.env   -- copy down right
```

To thread[13] a counter *uid* (unique identifier) through the tree, we use the following rules:

```
sem Expr prod App
  inh.left.uid  = id inh.lhs.uid     -- copy down to left
  inh.right.uid = id syn.left.uid    -- from left to right
  syn.lhs.uid   = id syn.right.uid   -- copy up from right
```

Similarly, if a production has one child, we may pass the value of an attribute of that child bottom-up as value for the same attribute of the parent:

```
sem Expr prod Lam   syn.lhs.errs = syn.b.errs   -- copy error messages up
```

Such *copy rules* [Magnusson et al., 2007] are so common[14] that we allow these rules to be omitted.

To make such an AG well-formed, the following algorithm augments an AG with copy rules. If a rule is missing for an inherited attribute *a* of a child *c*, we insert a rule that takes an attribute with the same name *a* from the local attributes, or synthesized attributes of the children to the left *c*, or the inherited attributes of the parent. The last attribute occurrence is taken in the ordering: inh from parent, syn of children, local attributes, inherited attrs of parent. In a similar way, we treat omitted copy rules for synthesized attributes. These copy rules can be considered to provide generic behavior for AGs that is not unlike the abstraction offered by reader and state monads.

Furthermore, we often collect attribute values in a bottom-up fashion:

```
sem Expr prod App   syn.lhs.errs = syn.left.errs ++ syn.right.errs
sem Expr prod Const syn.lhs.errs = []
```

Such *collection rules* [Magnusson et al., 2007] can also be inferred automatically when we specify a combination operator and an initial value:

```
attr Expr   syn errs use (++) []
```

This approach captures the abstraction provided by writer monads.

[13] When we write that we thread an attribute *x* through the tree, then we actually mean that we thread a value through the tree via a sequence of attributes that are all named *x*, and that are chained together by (mostly) copy rules. In a similar way, we talk about passing attributes topdown and bottom up.

[14] The AGs of UHC have more than twice as many copy rules than explicitly written rules. Thus the inference of copy rules saves a lot of manual labor.

Intermediate nodes and copy rules. It is often convenient to have to have intermediate nodes in the AST structure. An important benefit of copy rules is that it allows us to transparently add intermediate nodes to the AST.

As an example, consider function application in the lambda calculus. It is usually expressed as a binary expression in the abstract syntax:

$$\textbf{grammar}\, \mathit{Expr} \quad \textbf{prod}\, \mathit{App} \quad \textbf{nonterm}\, f, a : \mathit{Expr}$$

A function call with multiple arguments (e.g. $f\ a_1\ a_2$) is thus encoded as a sequence of applications $((f\ a_1)\ a_2)$ using *App*. Suppose that we want to add an inherited attribute that describes at which argument position an expression occurs:

$$\begin{array}{lll} \textbf{attr}\, \mathit{Expr} & \textbf{inh}\, \mathit{index} :: \mathit{Int} & \\ \textbf{sem}\, \mathit{Expr} & \textbf{prod}\, \mathit{App} & \mathit{inh.f.index} = \mathit{inh.lhs.index} \\ & & \mathit{inh.a.index} = 1 + \mathit{inh.lhs.index} \end{array}$$

This definition is incorrect for nested function calls (eg. $f\ a_1\ (g\ a_2)$), because the first argument a_2 of the nested call receives 2 as value for its *index* attribute. We get the expected behavior by distinguishing whether an expression occurs to the left or right of an application with an inherited attribute. This, however, requires us to specify this attribute for each occurrence of an expression nonterminal.

We obtain a more concise solution if we assume there is always a special top node above a sequence of applications:

$$\begin{array}{lll} \textbf{grammar}\, \mathit{Expr} & \textbf{prod}\, \mathit{AppTop} & \textbf{nonterm}\, e : \mathit{Expr} \\ \textbf{sem}\, \mathit{Expr} & \textbf{prod}\, \mathit{AppTop} & \mathit{inh.e.index} = 0 \end{array}$$

The copy rules transparently connect the remaining attributes of *e* with attributes of *lhs*. For these attributes, the existence of the intermediate node is not visible. A typical place to add these intermediate nodes is in the parser or with a tree transformation.

If we furthermore ensure that a special root node occurs above expression trees, then we also easily define an initial value for the *index* attribute:

$$\begin{array}{lll} \textbf{grammar}\, \mathit{ExprTop} & \textbf{prod}\, \mathit{Top} & \textbf{nonterm}\, e : \mathit{Expr} \\ \textbf{sem}\, \mathit{ExprTop} & \textbf{prod}\, \mathit{Top} & \mathit{inh.e.index} = 0 \end{array}$$

With nonterminal sets (Section 1.3.6) we can define attributes that are common to *Expr* and *ExprTop* without code duplication.

Higher-order children and copy rules. To improve the effectiveness of copy rules further, and in general to improve the separation of concerns, we can encode crosscutting rules as higher-order children (Section 1.3.7). By encoding a rule *r* as a child, we abstract *r* from how it is combined with other rules.

This idea is the spirit of this thesis. A higher-order child can be used to declaratively specify tasks (for instance, to ensure that two terms are equal), and the underlying implementation

```
grammar Unify prod Unify          -- nonterminal that represents unification
attr Unify  inh ty₁,ty₂ :: Ty     -- the two input types
            inh env     :: Env    -- the input environment
            chn subst   :: Subst  -- the input/output substitution
            syn errs    :: Errs   -- the output error messages
sem Unify prod Unify              -- essentially a wrapper around unify
  (lhs.errs, syn.lhs.subst) = unify lhs.env lhs.ty₁ lhs.ty₂ inh.lhs.subst
```

Figure 1.25: Encoding of *unify* as a higher-order child.

```
sem Expr prod App
  child u : Unify = sem_Unify              -- higher-order child
  u.ty₁          = f.ty                    -- input type
  u.ty₂          = (a.ty → loc.res)        -- input type
  inh.u.subst    = syn.a.subst             -- copy rule
  syn.lhs.subst  = syn.u.subst             -- copy rule
  u.env          = lhs.env                 -- copy rule
  lhs.errs       = f.errs ++ a.errs ++ u.errs  -- copy rule
```

Figure 1.26: The DHM rules for *App* with copy rules.

(the unification function) reflects the effects of performing the task in terms of attribute values. The orchestration of these tasks is determined by how the rules weave the attributes together.

In general, we can represent any function as a higher-order child. In Figure 1.25 we introduce a nonterminal for unification that has inherited attributes for each input of unification and synthesized attributes for each output of unification. The nonterminal has only one production, which contains only one rule, which is the rule we abstract over.

In a similar way we can encode the *fresh* and *instantiate* functions as higher-order children (Section 5.2.3). In Figure 1.26 we show how this is done for production *App*, where we added the unify-nonterminal as higher-order child *u* to the production, and added rules for the attributes of *u*. The latter rules are all implied by the copy rule mechanism and can be omitted.

Remarks. The automatic completion of an AG with copy rules is a double-edged sword. The mechanism saves a lot of boilerplate code, but the automatic behavior may not always be intended. If we accidentally forget to define an attribute explicitly, and that attribute can be given a default definition via a copy rule, then we are not warned that a rule is omitted. We provide a way to specify copy rules per production in Chapter 5, which gives more control over where and what copy rules are applied.

The copy rule mechanism uses the order of appearance of children for threading and bottom-up collections. This is sometimes not the appropriate order for a given application. For example, higher-order children follow conventional children in the order of appearance, thus are always at the end of copy rule chains. In such situations we can override the copy rule behavior by giving explicit rules for attributes. However, this reduces the convenience of copy rules, and the underlying idea that we rather specify patterns than individual rules.

As a solution, in Chapter 3 we allow the visit order to children to be explicitly specified for eagerly evaluated AGs, and can then use copy rules that use the visit-order instead of the order of appearance. Moreover, in Chapter 4 we define commutable (copy) rules, which are rules that can be ordered independently of their value dependencies.

1.3.13 Advantages and Disadvantages

The greatest advantage offered by AGs is that specifications are composable. Productions, attributes and rules can all be specified separately and automatically combined into a monolithic specification (Section 1.3.6). This easily allows new attributes and behavior to be added and shared with already existing rules. In particular, extra attributes can be used to specify additional administration for type inference strategies, and for the specification of what a compiler does with the inferred types.

In general, AGs offer modularity [Farrow et al., 1992] and extensibility [Viera et al., 2009], which can be realized via generic programming and meta programming approaches. In this thesis, we make use of such facilities, although these facilities themselves are not the focus of this thesis. Instead, we focus on combining AGs with algorithms that implement the functionality specified by declarative aspects of type rules.

1.4 Background on Ruler

This thesis complements previous work by Dijkstra and Swierstra [2006b] on the Ruler language and tool suite. Ruler gives a semantics to type rule descriptions in the form of an implementation with conventional attribute grammars and Haskell. Consequently, Ruler in its current state provides only an implementation for syntax-directed type rules.

The purpose of this thesis is to provide a core language *RulerCore* for Ruler which allows more complex inference strategies to be described. RulerCore extends on attribute grammars in various ways. Therefore, we focus more on attribute grammars than on type rule descriptions, although type systems play a prominent role in our work. The work on Ruler provides a notation for type rules and show how this notation translates to attribute grammars. From this work a notation that maps to RulerCore can be designed, hence we describe Ruler in this section.

```
relation expr view D                          -- declares the typing relation expr for D
  holes      g :: Gam   e :: Expr     t :: Ty  -- specifies parameters of expr
  judgespec  g ⊢ e : t                         -- notation for judgments of expr
relation member view D                        -- declares the relation member
  holes      g :: Gam   x :: String   t :: Ty  -- specifies parameters of member
  judgespec  (x,t) ∈ g                         -- notation for judgments of member
```

Figure 1.27: Examples of declaring relations in Ruler.

1.4.1 Ruler Features

Ruler aims to generate both a type system specification and a type inference implementation from a single description of the type system with type rules. The generated specification consists of type rules formatted to LaTeX figures, and can be used as documentation and for formal reasoning. The generated implementation consists of attribute grammars that can be included verbatim in the source code of a compiler. This approach guarantees consistency between the specification and the implementation.

Ruler provides notation for the incremental description of type rules. It features the addition of parameters to judgements, type rules to relations, and judgements to type rules in a similar way as we can add attributes, productions, and rules to an AG description.

When type rules are declarative, only well-formedness checks and generation of the specification is possible. However, Ruler provides also notation to describe algorithmic rules. Using the facilities for incremental descriptions, a direction can be given to the parameters of typing relations, and type rules can be associated with productions of an accompanying attribute grammar, which turns the typing relation in a deterministic function for which AG code can be generated.

1.4.2 Ruler Concepts

We take the explicitly typed lambda calculus as described in Section 1.2.3 as basis to show the main concepts and syntax[15] of Ruler. In Ruler, a type system description is a composition of views on relations and their rules.

Definition (View)**.** In Ruler, a *view* is a named subset of the declared relations, holes, judgments, rules, etc. Each view describes a type system.

We start with a declarative view, which we give the name D. Later, we provide also an algorithmic view with the name A.

Definition (Hole)**.** In Ruler, the parameters of a relation are called *holes*. The parameters are explicitly named and are explicitly typed.

[15] We took the freedom to deviate slightly from Ruler's actual syntax in order to have closer correspondence with the notation that we use in this thesis.

```
ruleset theRules relation expr         -- a set of rules for expr
  rule e.var view D                    -- rule for the var-case
    judge L : member   (x,t) ∈ g       -- premise with the name L
    —
    judge R : expr     g ⊢ x : t       -- conclusion with the name R
  rule e.app view D                    -- rule for the app-case
    judge F : expr     g ⊢ f : t1 → t2 -- premise with the name F
    judge A : expr     g ⊢ a : t1      -- premise with the name A
    —
    judge R : expr                     -- alternative syntax for judgment
      | g = g                          -- binds expression g to hole g
      | e = f a                        -- binds f a to hole e
      | t = t2                         -- binds t2 to hole t
```

Figure 1.28: Example of Ruler rules.

Figure 1.27 shows how to declare a 3-place relation *expr*. The line with *judgespec* specifies a custom notation (meta-grammar) for the *expr* relation in terms of a meta grammar production. Each hole must be uniquely present as meta nonterminal in this meta production.

Definition (Ruleset)**.** A *ruleset* is a group of rules in combination with a collection of meta-information, such as a name for the group.

A relation is either a foreign relation, or it is specified by the rules of some ruleset. Figure 1.28 shows the ruleset *theRules*. A rule is given an explicit name, and zero or more judgments as premises above the line, and one premise under the line. A judgment has an explicit name and corresponds to a relation. The arguments are either bound to the corresponding holes via the custom syntax in a nameless way, or via a generic syntax where each binding is explicitly given.

The language of Ruler expressions consists of meta variables (such as f and a), and external constants and operators (such as the function arrow). The interpretation of such terms depends on the target language. Denotations can be given for individual symbols and constants, as well as for (saturated) applications:

```
rewrite  ag ((t1 :: Ty) → (t2 :: Ty)) = (TyArr t1 t2) :: Ty   -- denotation of application
external TyArr                                                -- identity denotation
rewrite tex ((t1 :: Ty) → (t2 :: Ty)) = (t1 → t2) :: Ty       -- fully saturated appl only
format  tex  →                        = "\rightarrow"         -- denotation of a symbol
```

Rewrite rules are applied in a bottom-up fashion. The LHS of a rewrite rule specifies a typed pattern to match against a Ruler expression. The RHS must again be a Ruler expression, which then is assumed to have the given explicit type. These types allow the notation to be overloaded. A rewrite rule applies if both the syntax and the types of the LHS matches the actual Ruler expression.

An explicit denotation must be given for foreign relations. The distinction between conventional relations and foreign relations is that the latter is not defined by rules. An explicit denotation to the host language needs to be given for a foreign relation:

```
relation member view D                  -- member is not specified by rules
  judgeuse tex (x,t) ∈ g               -- denotation for LaTeX
  judgeuse ag  (Just t) = lookup x g   -- denotation for AGs with Haskell
format tex ∈  = "\in"                   -- denotation of a symbol
external Just lookup                    -- identity denotation
```

The above declarative specification can be typeset to LaTeX. To describe an algorithmic version, additional information needs to be added to the relations and rules. We can describe these additions separately by defining a view *A* that extends from *D*:

```
viewhierarchy D < A   -- partial order on views
```

In particular, we need to define how the relations are mapped onto nonterminals of an AG, and turn the relations into deterministic functions such that the parameters can be mapped to attributes.

Ruler code does not stand on its own. The generated code for type inferencing is supposed to be used in conjunction with other AG code. An association is specified between relations (schemes) in Ruler and nonterminals in the AG. For example, the relation *expr* is associated with the nonterminal *Expr*. Its type rules are associated with productions of *Expr*. This correspondence is made explicit by annotating the nonterminal declaration with the name of the relation, and the productions with the name of the corresponding rule[16]:

```
grammar Expr [expr] view A   -- relation expr mapped to nonterminal Expr
  prod Var [e.var]           -- rule e.var mapped to production Var
    term nm :: String
  prod App [e.app]           -- rule e.app mapped to production App
    nonterm f : Expr
    nonterm a : Expr
```

Moreover, the holes are mapped attributes. Judgments represent the semantics of a production. Judgments of a foreign relation are translated to AG rules. The other judgements correspond to children of the production and are mapped to rules that define the inherited attributes of the children, or pattern match against the synthesized attributes.

In view *A*, we refine the declaration of *expr* to include additional information. In this example, we only provide additional information about the attributes. In general also additional attributes and syntax can be added. We map the holes of the *expr* relation to inherited or synthesized attributes. The hole that corresponds to the AST gets the special *node* designation:

[16] Ruler allows productions to be defined by combining rules. Within the square brackets may not only be a name of a rule, but also an expression that denotes a composition of rules. An example is the left-biased union of two rules.

```
ruleset theRules relation expr                  -- extensions to rules for expr
  rule e.var view A                             -- rule for the var-case
    —
    judge R : expr   g ⊢ (node nm = x) : t      -- association between term nm and x
  rule e.app view A                             -- rule for the app-case
    —
    judge R : expr                              -- convenient for extensions
      | e = (node f = f) (node a = a)           -- assoc child f to f and child a to a
```

Figure 1.29: Rules demonstrating node-holes.

```
relation expr view A      -- algorithmic view on expr
  holes node e :: Expr    -- AST node
        inh g  :: Gam     -- input param is inherited attribute
        syn t  :: Ty      -- output param is synthesized attribute
```

A Ruler expression is at a defining position if it is bound to an inherited hole of a premise, or bound to a synthesized hole of a conclusion judgment. Otherwise, it is at a using position. The generated algorithm constructs a value at defining positions, and matches against values at usage positions.

A node-hole takes the AST as value. In judgments of rules, node holes are treated differently with respect to normal holes in order to define the correspondence between judgments of the type rule and children of the production. The node hole of a premise judgment may only be an identifier and corresponds uniquely to child of the associated production. In the expression bound to the node hole of the conclusion, these identifiers must occur and we annotate them with the name of the child of the production. Figure 1.29 shows an example. The syntax for explicitly named bindings of holes is convenient for extensions, as only the bindings that are redefined need to be mentioned.

When a Ruler description is well-formed, an AG can be generated from the above description. This AG contains attributes and semantics for type inference, and can be combined with other attributes and Haskell infrastructure into a compiler.

1.4.3 Damas-Hindley-Milner Inference

Similar to Section 1.2.6 and Section 1.3.11 we show in this section a DHM inference algorithm in Ruler. In the DHM view, we add lambda abstractions without an explicit type annotation and a let expression. However, we only show the code for the app-rule, since we already explained Ruler's concepts in the previous section.

In Figure 1.30 we add a DHM view on top of the algorithmic view, and define external relations to obtain fresh types and to unify two types. The retain-keyword declares that the left component of the output tuple of *unify* is mapped to a local AG attribute *loc.errs*, which

```
viewhierarchy D < A < H                       -- DHM view H
relation expr view H                          -- DHM view on expr
   holes chn s  :: Subst                      -- substitution as threaded attribute
relation tyFresh view H                       -- wrapper for fresh
   holes chn s  :: Subst                      -- threaded subst
          syn t  :: Ty                        -- fresh type
   judgespec    t ‘fresh‘                     -- syntax for the judgement
   judgeuse ag (t, syn.s) = fresh inh.s       -- denotation for AGs with Haskell
relation tyUnify view H                       -- wrapper for unify
   holes chn s  :: Subst                      -- threaded subst
          inh  t1 :: Ty1                      -- left type
          inh  t2 :: Ty2                      -- right type
   judgespec    t1 ≡ t2                       -- syntax for the judgment
   judgeuse ag (retain errs, syn.s) = unify g t1 t2 inh.s
```

Figure 1.30: DHM relations in Ruler.

can then be collected by conventional AG rules. This mechanism allows values to be exposed as attributes to the encapsulating AG.

Figure 1.31 shows the rules for *e.app* in view H. The hole-bindings for judgments essentially specify the threading of the substitution. The notation for hole-binding can be used to supply bindings for holes that are not present in the judgement's special syntax. To connect two nodes, we introduce an intermediate meta variables s_1, s_2, etc. This threading has to be done manually as Ruler does not have a concept of copy rules.

1.4.4 Discussion

Ruler provides notation and composition mechanisms for the description of type rules. Rules may inherit from other rules, and rules inherited from the same rule from preceding views. These mechanisms enhance modularity and reuse. Effectively, a ruler fragment is a partial specification. The meaning of a ruler specification is only defined for a complete composition in combination with the associated attribute grammar. It would aid formal reasoning if a meaning can be attached to individual fragments.

The Ruler compiler generates only an inference algorithm for algorithmic specifications. As a consequence, the code generation is limited to syntax-directed type rules. Syntax-directedness does not hold for many declarative type systems that have relations with overlapping rules, or rules that dispatch on more than one argument of the conclusion judgment. Also, the premisses must be functional. The transformation of a relation to a function by itself is non-trivial, and common techniques such as fixpoint iteration or search strategies are not directly supported by Ruler. We address these complications in this thesis (Section 1.5).

Moreover, Ruler's features are actually not specific to the domain of type systems. Ruler

```
ruleset theRules relation expr
  rule e.app view H
    judge T : tyFresh r ‘fresh‘
      | inh.s = s1      | syn.s = s2
    judge F : expr
      | t     = t1      | inh.s = s2  | syn.s = s3
    judge A : expr
      | t     = t2      | inh.s = s3  | syn.s = s4
    judge U : tyUnify t1 (t2 → r)
                        | inh.s = s4  | syn.s = s5
    —
    judge R : expr      | inh.s = s1  | syn.s = s5
```

Figure 1.31: Ruler rules for *e.app*

provides a rudimentary composition mechanism, syntactic sugar, and type setting support for a formalism that is not unlike AGs. These features would equally well benefit AGs in general[17].

1.5 Thesis Overview

Our ultimate goal is to semi-automatically generate a type inference algorithm from a declarative type system specification. In particular, we focus on type systems described as a collection of type rules, and an implementation based on attribute grammars.

In this thesis, we present RulerCore, a language that extends attribute grammars. It facilitates the composable description of inference algorithms that are typically used to implement declarative aspects of type rules. RulerCore can thus be used to give an executable semantics to a set of type rules.

In this section, we give an overview of RulerCore's extensions on attribute grammars. In Section 1.6 we position this thesis with respect to the larger goal.

1.5.1 Inference Algorithms as an Attribute Grammar

We give an abstract description of how we structure inference algorithms as an AG. Chapter 5 shows a concrete example.

We give AGs over typing derivations, instead of AGs over the abstract syntax of a language. In the following chapters of this thesis, we refer with *abstract syntax tree* either to typing

[17] An example of a feature that benefits AGs is Ruler's automatic unique numbering mechanism. We generalized and implemented a similar mechanism for AGs.

derivations or to the result of parsing[18]. Nonterminals thus correspond loosely to typing relations, and productions to type rules (Section 1.3.10). The grammar for typing derivations may have more structure than is present in the type rules. For example, we may add nodes that each represents a choice between alternatives, so that we effectively describe a forest of typing derivations.

We map each declarative aspect of a type rule to a higher-order child (Section 1.3.7) as shown by Section 1.3.12. Extra attributes, such as a substitution, provide contextual information for these children. For example, an equality premise between two types in the type rules corresponds to a unification-child in the AG. The structure of the unification child servers as proof that the two types can be made equal, and the resulting substitution attribute reflects the effect of unifying the two types. We treat meta variables in type rules as conventional attribute values (Section 1.3.11).

Type inference amounts to determining the structure of these children. To choose a particular tree for a child may require an exploration of candidate trees. Instead of constructing a single derivation tree, we actually construct and choose from a forest of derivation trees. Through value dependencies between attributes, we effectively define in which order the structure of these children is determined, and in which order the effects of determining this structure is visible in the substitution attribute.

This approach allows us to encode Algorithm W (Section 1.2.6). For more complex inference algorithms, such as constrained-based algorithms and algorithms that require fixpoint iteration, the shape of the derivation tree and the values of attributes are mutually dependent. Inference algorithms therefore analyze, explore, and extend (intermediate) partial derivation trees. This process does not have a straightforward mapping to an AG, since an AG specifies when the resulting derivation tree is correct, but not how the intermediate trees are obtained. We introduce extensions to AGs to make such intermediate trees visible in the AG description.

1.5.2 Attribute Grammar Extensions

Chapter 2 gives a detailed outline of each extension. The following chapters work out each extension individually.

Visits. RulerCore is a language for the description of higher-order, ordered attribute grammars. We extend this basis with explicit specifications of visits. In comparison, visits are implicit in ordered attribute grammars. Chapter 3 introduces the language and the notation.

For ordered attribute grammars, there exists an evaluation algorithm that starts with an initially undecorated tree and ends in a correctly decorated tree. The *state* of a tree is the collection of decorations present in the tree. During the evaluation, the state of the tree thus changes. A *configuration* is a set of attribute names. A configuration describes the state of a tree when the set of decorations of the root contain exactly the attributes as mentioned in the configuration. In an ordered attribute grammar a linear order exists between configurations.

[18] The typing derivation is typically an extension of the AST, thus the difference is usually irrelevant and can be determined from the context. Similarly, we refer to AST and the AST decorated with attributes interchangeably.

A *visit*, a unit of evaluation for a node, transitions the state of a tree to a state described by the next configuration. We treat a node with a state described by a certain configuration as a *first class value*, which we can store in attributes, obtain from attributes, inspect, and programmatically apply state transitions on. This approach offers us sufficient control over the AG evaluation to combine AGs with *monadic operations*, such as the unification monad as shown in this chapter.

We provide notation to declare a totally ordered sequence of visits per nonterminal. Each attribute declaration must be associated with one visit declaration, which has consequences for the scheduling of rules. During attribute evaluation, attributes that are associated with an earlier visits are defined before attributes of a later visit are computed/defined. In this context *defined* means that a reference to the attribute value is available. Similarly, rules of productions of a nonterminal may be associated with a visit of that nonterminal, which restricts the scheduling of such rules to either that visit or to a later visit. Furthermore, we may restrict a rule to a particular visit, which ensures that the rule is scheduled before any rules of subsequent visits.

With this approach, rules may make assumptions about the configuration of tree tree prior to the rule's evaluation. Moreover, the notation allows us to define customizations of evaluation strategies for rules of a particular visit or for visits to particular subtrees.

A disadvantage of our approach with respect to attribute grammars is that we need to specify a visit for each attribute. This requires additional effort and makes attribute declarations less composable. We typically declare an attribute for a set of nonterminals instead of a single nonterminal. In our approach, this is only possible if all the nonterminals in the set have a common visit to which the attribute can be scheduled. We show in Chapter 4 how to solve this issue.

Fixpoint iteration, clauses and constraints. In Chapter 5 we exploit the notion of visits. We show how we to conditionally repeat the evaluation of a visit to a child, which allows the encoding of *fixpoint iteration*. Using visit-local attributes, a state can be kept between iterations. Moreover, we allow one or more clauses to be defined for a visit. Each clause provides an alternative set of rules for the visit. With special match-rules we specify constraints on clauses. With clauses in combination with higher-order children we can define the structure of the derivation tree in terms of attributes, and thus deal with type-directed inference algorithms.

We treat intermediate derivation trees as first class values. With the specification of visits, we can reason about the configuration a tree is in. A tree that is in a certain configuration can be detached, transfered via attributes to another location, and attached there. With this mechanism we can represent *constraints* or *deferred judgments* as trees with access to their context via attributes.

Exploration of alternatives. The above extensions describe algorithms that conservatively approximate the derivation tree. For some type systems it is necessary to explore a forest of candidate derivation trees. Such a forest can be represented with a decision tree, which contains choice nodes that branch to various *alternatives*.

In Chapter 7 we show how to describe explorations of such alternatives with AGs. We present a technique that allows a spectrum of depth-first and breadth-first search strategies to be described. In a statically scheduled AG, we can evaluate the AG in a step-by-step fashion. By intertwining the evaluation of alternatives, we obtain a breadth-first search. After each step, some intermediate values may be available, which can be used to direct the search process.

Phases and Commuting Rules. In Chapter 4 we generalize visits to *phases*. A phase may consist of one or more implicitly defined visits, which are determined by the static scheduling of the AG. As a consequence, an attribute does not need to be explicitly assigned to a visit, and its scheduling may optionally be constrained by a phase. Using this approach, our extensions extend AGs conservatively.

For some chained attributes, the order induced by value dependencies of their rules may be too strict when these rules encode commuting operations. We present *commuting rules*, which are rules that are connected via a chained attribute, but which do not depend on previous rules in the chain, Such rules thus give us more freedom in the scheduling of these rules. Typical examples are the threading of a unique number supply, and the threading of substitutions. To preserve referential transparency, the commuting rules must satisfy a liberal commutativity law. With such rules we can functionally encode the behavior of a rule with side effect that is scheduled to different implicit visits.

Dependent AGs. In Chapter 9 we apply dependent types to AGs. In a dependently typed AG, the type of an attribute may refer to values of attributes. The type of an attribute is an invariant, the value of an attribute a proof for that invariant. Thus, with dependent AGs we can proof properties of our compiler. Additionally, this chapter serves as a showcase for visits and clauses.

1.5.3 Contextual Chapters

In the extended edition of this thesis[19], we place the above extensions in a wider context.

Graph Traversals. Many computations in a compiler take control-flow or data-flow graphs into account. We show that the mechanism to attach and detach children can be used to interface AGs with graph traversals [Middelkoop, 2011b].

GADTs. In the extended edition [Middelkoop, 2011a], we show an example of a type system for Generalized Algebraic Data Types (GADTs). An inference algorithm for this system requires an exploration of alternatives (Chapter 7). We formulate our specification so that it is orthogonal to specifications of ADTs:

- We present our specification as System F augmented with first-class equality proofs.

[19] Extended edition: `https://svn.science.uu.nl/repos/project.ruler.papers/archive/thesis-extended.pdf`

- We exploit the Church encoding of data types to describe GADT matches in terms of conventional lambda abstractions.

Such orthogonal designs are important in order to compose type systems, and ultimately thus also to compose type inference algorithms.

1.6 The Context of this Thesis

Types play an increasingly more important role in the design of programming languages. Type systems specify a relation between programs and types, which facilitates (formal) reasoning with typed programs. Moreover, type systems form a partial specification for type checking and type inference algorithms. As we discussed in Section 1.1, our ultimate goal is to semi-automatically derive type inference algorithms from declarative type system specifications. We mentioned in Section 1.2 that a set of type rules alone is not a complete description, hence we develop Ruler (Section 1.4), which is a domain-specific programming language in which we write an inference algorithm as an extension of the declarative type rules.

1.6.1 Challenges

There are several challenges that need to be overcome to reach this goal. We identify two main challenges. This thesis is situated in the second challenge.

The first challenge is related to type system compositions. Language features and their declarative type systems are typically defined as extensions of a bare lambda calculus. Some language features are mutually conflicting (e.g. invalidate type soundness). However, many language features compose in standard ways. For example, features described for a lambda calculus that support fix-expressions can be translated to a description for a language with recursive let bindings.

To meet this challenge, we wish to describe language features in isolation, and describe a composition of these features for the actual source language. As illustration, the type rule formalism lacks the expressiveness that higher-order functions offer to functional programs, such as the ability to abstract over common patterns, and to instantiate these abstractions with minimal syntax. For small type system descriptions that appear in type system theory, such expressiveness is not needed. However, type system descriptions of actual languages are large and hard to maintain.

The second challenge is related to declarative type rules. Declarative type rules abstract from evaluation strategies. However, a general inference algorithm does not exist, and a naive algorithm such as mentioned in Section 1.2.4 is either incomplete or inefficient.

Ultimately, inference boils down to resolving *declarative aspects*: to determine the structure of the derivation tree, and computing with values that may not be fully determined yet. In practice, inference algorithms are intricate compositions of common algorithms that treat such declarative aspects in a predictable and deterministic way. These algorithms are hard to combine. If one declarative aspect requires a constraint-based algorithm for its resolution, and another requires some form of search, then the order in which the aspects are resolved is likely to be relevant. Also, it is hard to describe the flow of information between different solving techniques. To allow these techniques to mutually cooperate, we need a language for the description of a composition of such techniques. Hence, this thesis.

1.6.2 Additional Challenges

Aside from the general motivation of our research, another source of motivation is that we intend the results of our research to benefit the implementation of our Haskell compiler UHC [Dijkstra et al., 2009]. Therefore, we impose additional demands on solutions to the above challenges.

Firstly, since UHC's implementation is based on Haskell itself, we require that solutions integrate seamlessly with Haskell. This restriction effectively rules out the direct use of (functional) logic languages, due to differences in the evaluation model and the representation of data structures[20]. In addition, we desire that our research can also be exploited in compiler suites that are implemented with languages without lazy evaluation or strict typing disciplines.

Secondly, along similar lines, we refrain from the use of dependently-typed languages, since an extraction to Haskell is a one-way process that also affects data-type representations. Our goal is to be able to generate an implementation. Advances in dependently-typed languages seem promising, but a formally certified implementation of a compiler such as UHC is currently infeasible.

Finally, the resulting implementation should be reasonably efficient in order to process ASTs of large programs. In our experiences with UHC, we noticed that memory usage is an issue when using demand-driven evaluation of AGs. We experimentally verified that the time spend on traversing abstract syntax trees in UHC is negligible in comparison to the computations that are performed on each node of the AST. Thus, while traversal overhead is rarely a problem, memory usage is an item of concern, which we address by using statically ordered evaluation of AGs.

1.6.3 Solutions

A partial solution to the first challenge is given by Dijkstra [2005], as demonstrated by the UHC project, and the initial development of Ruler (Section 1.4) in particular.

Ruler's composition mechanisms and syntax extensions that are provided by Ruler would be beneficial to AGs. For example, many dense translation schemes in this thesis are manually derived from actual AG descriptions. These AG descriptions focus on attributes in isolation and are easier to understand, but too verbose for inclusion in this thesis. Solutions for AGs would also work for type rule descriptions, and vice versa. Indeed, composition facilities for AGs receive ongoing attention [Viera et al., 2009, Saraiva, 2002].

[20] Braßel et al. [2010] show that an embedding of functional logic programs is possible in Haskell, but affects all data representations and forces all computations to monadic style. However, we use techniques techniques and ideas from logic programming that integrate seamlessly, such as backtracking in a monad [Hinze, 2000, Kiselyov et al., 2005].

There is still a long way to go with respect to the first challenge. Since declarative rules abstract from an evaluation algorithm, the data structures and administration that are involved in the algorithm are chosen for convenience and notational conciseness. In an actual implementation, we may be able to represent certain administration in a more efficient way using specialized data structures. However, we do not address these issues in this thesis, and only mention some in passing in Middelkoop [2011a]. We assume that the declarative rules are specially crafted to make them more suitable for an actual implementation.

There is thus some open work for the first challenge. However we focus on the second challenge. This challenge is more pressing, because we need basic building blocks before we can compose them in clever ways.

We propose to tackle the second challenge with attribute grammars. Inference algorithms that are specified by declarative type rules are sensitive to context—non-inductive properties of the AST—and attribute grammars excel in providing such contextual information with attributes, as is proven by UHC's implementation. Also, from a practical perspective, since UHC's implementation is based on AGs, an inference algorithm based on AGs interfaces conveniently with attributes of other components in the compiler, as shown by previous work on Ruler.

However, attribute grammars in current form at not well suited for the description for inference algorithms of complex type systems. Inference algorithms therefore make explicit assumptions about the intermediate states of the derivation trees during its construction. In a conventional AG, we cannot do so, because AG descriptions are defined in terms of the final state of the derivation tree. In order to make assumptions about the intermediate state, we extend AG evaluation and hence arrive at Section 1.5.

Our approach applies to the description of algorithms that are recursive functions over tree-like data structures. In particular, our approach applies to catamorphisms, which is not surprising because attribute grammars can be considered a domain specific language for the description of catamorphisms. On the other hand, for example, algorithms based on graph rewriting are not straightforwardly expressed in our approach. An inherent difference is that we traverse a structure whereas rewrite rules as used by graph rewriting access the structure in irregular ways. Also, algorithms that involve matrix operations to efficiently solve linear constraints cannot be described straightforwardly. However, Middelkoop [2011b] shows how to mix attribute evaluation with external solvers.

1.7 Related Work

Each chapter has its own related work section. In this section we consider work that is related to the thesis as a whole.

1.7.1 Circular AGs and Exposure of Intermediate States

Evaluation algorithms for circular AGs [Jones, 1990, Magnusson and Hedin, 2007] provide an alternative way to extend the AG evaluation algorithm. For circular AGs, the algorithm is parametrized with an initial value for cyclic attributes. The algorithm describes a repeated attribute evaluation to compute a fixpoint for the cyclic attributes. When used in the context of this thesis, such attributes thus expose intermediate states of the derivation tree during evaluation.

Fixpoint iteration is one of many evaluation strategies (Section 1.2.4). For example, several type systems use an algorithm that describes the exploration of multiple candidate derivation trees (Chapter 7). Moreover, to expose the intermediate states of the derivation tree as an attribute, attributes at various locations of the tree have to be explicitly stored into and obtained from this attribute[21], which is cumbersome and destroys modularity. Our extensions generalize over fixpoint iteration and many other common techniques employed by inference algorithms.

1.7.2 Proof Assistants

Proof assistants such as Isabelle/HOL [Wenzel et al., 2008], Twelf [Schürmann, 2009], and Sparkle [Mol et al., 2002] and Coq [Bertot, 2006] can be used to formalize a type system and inference algorithm, and prove various consistency properties between specification and implementation. Typically, the formalized inference algorithm can be extracted to code in some target language. Such an approach is possible for small type systems as encountered in theory, but does not scale up when dealing with practical, full-blown type systems of large languages[22].

In Coq, but also in other dependently typed languages such as Agda [Norell, 2009], Epigram [McBride, 2004], and IDRIS [Brady, 2011], properties of type system can be expressed as types of the inference algorithm. To do so, the inference algorithm needs to be implemented, and structured so that it can be complemented with proofs of properties, such as type soundness. In this thesis, we consider type systems for which the first task is already difficult, and the second task infeasible in practice. Thus, such approaches are out of the scope of this thesis. However, we consider dependently typed languages in Chapter 9.

Closer to the goals expressed in this thesis are the languages Ott (Section 1.7.3) and TinkerType (Section 1.7.4).

1.7.3 Ott

Ott [Sewell et al., 2007] and SASyLF [Aldrich et al., 2008b] are meta languages in which formal semantics such as type systems can be formalized. Similar to Ruler [Dijkstra and Swierstra, 2006b], these languages provide special syntax for inference rules, and require the rules to be well-formed. In contrast to Ruler, the purpose of these languages is to aid the construction of proofs. From an Ott-description, boilerplate code for several proof assistants (e.g., Coq) can be generated. This boilerplate code consists of parsers for concrete syntax, abstract syntax, and substitution lemmas.

[21] Such an attribute models a heap in a similar way as a substitution models memory.
[22] As demonstrated by Faxén [2002], a type system for Haskell'98 is already large and complex.

Concrete and Abstract Syntax. The following is specification of a grammar for a variant of the lambda calculus with tuples in Ott[23]. Such a grammar consists of three types of symbols: meta variables, nonterminals and terminals. Meta variables are nonterminals that can be substituted and alpha-renamed. Nonterminals are introduced by the grammar. The remaining symbols that occur in the grammar are considered terminals. Occurrences of a nonterminal may take a subscript to distinguish multiple occurrences of the same nonterminal from each other.

```
metavar x
grammar  e ::=                  :: Expr
           |    x               :: Var
           |    \ p . e         :: Lam :: bind b(p) in e
           |    e1 e2           :: App
           |    ( e )           :: _

         p ::=                  :: Pat
           |    x               :: Var :: b = {x}
           |    (p1, ..., p.n)  :: Tup :: b = b(p1) ... b(p.n)
```

The grammar specifies a concrete syntax to the left of the double colon, and the constructor for the abstract syntax to the right. The production for parentheses is considered a meta-production and is not reflected in the abstract syntax.

An important concept of Ott is *binding*. After the second double colon, *binders* for meta variables can be specified, as well as their scope. The expression `b(p)` represents the set of meta variables defined by the synthesized attribute `b` of `p`. Zero or more synthesized attributes may be specified for a nonterminal. The binder `bind b(p) in e` for the lambda production denotes that the meta variables `b(p)` are bound at this lambda, and are in scope of *e*. Lemmas for substitution and alpha equivalence, as well as a definition of free variables are derived from the binder annotations. The underlying mechanism ensures through alpha renaming that meta variables are not accidentally captured by substitutions.

Ott has a notion of list forms, which is convenient syntax to represent the common over-bar notation that is used grammars and type rules. Triple dots can be used to construct list patterns, list expressions, and lists of judgments. For example `b(p1) ... b(p.n)` is a list expression, where `n` is an index variable. Also, list comprehensions and projections of list items can be used.

Inference Rules. Judgements can refer to relations defined in Ott via type rules, or to externally defined functions and relations in the target language. For example, to model a call-by-value operational semantics, the following code fragment represents beta reduction.

```
defn  e1 --> e2 :: reduce


                  isValue e2
```

[23] For presentation purposes, the examples use a slightly different notation than provided by Ott.

```
  ---------------------------------------- :: call-by-value beta
           (\x.e1) e2  -->  {e2/x} e1
```

The defn line specifies the syntax of the reduce judgments, and is followed by the inference rules for that relation. The premise judgments occur above the horizontal line and conclusion judgments below. The infrastructure for the substitution e2/x is derived from the binder specification of e1. The inference rules are translated to axioms in the target language for the reduction relation.

Discussion. Approaches such as Ott aid formal reasoning about type systems, and thus pursuit a different goal than the derivation of type-inference implementations from specifications. On the other hand, Ott and Ruler share the common goal of formalizing type systems and specifying properties. Certain concepts are also beneficial to AGs. Binding and scoping is very common, so the concept of binding may be very useful for AGs as an abstraction for name analysis. Similarly, list forms would benefit AGs when using higher-order children (Section 1.3.7).

1.7.4 TinkerType

TinkerType [Levin and Pierce, 2003] is a language for the modular description of whole families of formal systems, with a focus on type systems and operational semantics. A type system is described in two ways. A system is described intensionally as a set of *features*. These features are names for abstract properties of a type system. A system is described intentionally as a set of *clauses*. In TinkerType, a clause is a denotation of a type rule in the form of LaTeX text or ML code.

Overview. A TinkerType description consists of a number of elements: features, dependencies between features, clauses, a clause refinement relation, and feature constraint formulas. With the latter two elements, clause refinement and feature constraints, a partial consistency between type systems can be expressed.

Distinct type systems have different clauses. However, type systems with similar features tend to have similar clauses. The relation between clauses is exploited by TinkerType. A TinkerType description therefore contains a whole repository of named clauses that are tagged with a number of feature names. Clauses with different feature sets can have the same name. Given a number of features, a type system is then assembled by the TinkerType Assembler by selecting the clauses that support these features best, which are the clauses where the provided feature set is a subset of the requested feature set. Duplicately tagged clauses are filtered out. The clause with the largest feature set is retained.

A dependency relation must be specified between features. A type system is fully defined by the transitive closure of the dependency relation on the set of features of the type system. Some combinations of features give an unsound type system, or the inference algorithm is incomplete. Certain combinations of features can be declared as deficient using feature constraint formulas. These are propositional formulas over features that must be satisfied with the features mapped to truth-values based on their presence in the type system. The dependency relation between features is one form of such a feature constraint formula, in the form of an implication.

Code Assembly. The code repository consists of *components*, which represents several clauses and support code. For example, the following fragment [Levin and Pierce, 2003] contains ML code that deals with the type checking of conditional and boolean expressions.

```
component bool, typing {
  parsing { ... }
  ast { ... }
  core {
    typeof {
      header {# let rec typeof gam t = match t with #}
      separator {#| #}

      T-If
        {#TmIf(e1,e2,e3) ->
          if equiv gam (typeof gam e1} TyBool
          then let res = typeof gam e2 in
            if equiv res (typeof gam e3)
            then res
            else error "branches differ in type"
          else error "guard is not a boolean"#}
} } }
```

The level of granularity of features is actually per component instead of per clause. A component consists of several sections related to parsing, abstract syntax tree representation, and the actual typing relations with their clauses.

The assembling process is essentially based on concatenating and substituting strings. The components that match the requested features are merged, and the produced code consists of the `header` followed by the clauses in verbatim, which are separated by the `separator`.

Consider a component with subtyping `sub` as additional feature:

```
component bool, typing, sub {
  core {
    typeof {
      T-If
        {#TmIf(e1,e2,e3) ->
          if [[subtype]] gam (typeof gam e1} TyBool
          then [[join (typeof gam e2) (typeof gam e3)]]
          else error "guard is not a boolean"#}
} } }
```

In another component with feature `sub`, the functions `subtype` and `join` are defined, which can thus be used in the above component.

The refinement relation between clauses is expressed by means of double bracket annotations in the source code. The fragments inside the double brackets are considered new, the fragments outside the double brackets must occur verbatim in the refined clause.

Discussion. TinkerType is modular in the sense that all clauses can be written separately, and the system enforces that clauses are designed with reuse of concepts in mind. A clause can only be reused verbatim. In practice, additional material needs to be added to a clause, for example, due to extra judgments or due to extra parameters to the judgments when supporting extra features. This leads to code duplication in the clauses with the usual engineering problems as a consequence. The static checks on clause refinement, however, are likely to catch errors resulting from incomplete code modifications, and encourage writing clauses as increments of each other.

As discussed above, some forms of consistency are expressed between clauses of different type systems, which is enforced by means of consistency checks that point to errors in the code. To make these checks effective, there must be a lot of overlap between clauses, and thus an extensive set of features with a fine granularity. Consistency is not expressed between clauses of the same type system. There is no guarantee that the collection of clauses results in compilable ML code or LaTeX text, nor that the ML code is in any way related to the LaTeX text.

1.7.5 Overview of Recent Attribute Grammar Systems

An in-depth exploration of AG systems is out of the scope of this thesis. We give some of the distinguishing features of current AG systems. Most current AG systems support a wide range of features including higher-order attributes [Vogt et al., 1989] and collection attributes [Magnusson et al., 2007].

The Lrc [Kuiper and Saraiva, 1998] is one of the few current AG systems that is based on ordered attribute evaluation. Its distinguishing feature is incremental evaluation. It has been used as vehicle for research in parallel evaluation and the generation of interactive programming environments. Lrc generates Haskell and C code, although it is not actively maintained anymore.

UUAG [Löh et al., 1998] can be regarded as a simplified reimplementation of Lrc, starting originally by piggybacking heavily Haskells lazy evaluation. Currently, UUAG supports ordered and demand-driven evaluation. Features such as incremental and parallel evaluation are being investigated, as well as first-class attribute grammars [Viera et al., 2009].

The AG systems that we consider below are based on demand-driven attribute evaluation. These systems support reference attributes [Magnusson and Hedin, 2007] which allow attributes to be defined and accessed from non-local nodes. Data-flow analyses with circular attributes are an application of reference attributes [Farrow, 1986].

Silver [Wyk et al., 2008] supports forwarding [Wyk et al., 2002] as distinguishing feature. Forwarding is a convenient notation for desugaring with higher-order children (Section 1.3.7) in combination with specialized copy rules (Section 1.3.12). With first-class AGs, Viera et al. [2009] implement a more advanced form of forwarding.

Silver also supports the specification of a (control-flow) graph structure on top of the AST [Wyk and Krishnan, 2007]. A production may specify CTL formula which are checked against the graph structure. This way, control-flow analyses can be implemented conveniently in attribute grammars (see also Middelkoop [2011b]).

JastAdd [Ekman and Hedin, 2007] has rewrite rules as distinguishing feature. Rewrite rules are applied to a tree upon the first access through demand-driven evaluation and can conditionally depend on attribute values.

1.8 Conclusion

In the following chapters, we present several extensions to attribute grammars that facilitate the description of complex type inference algorithms. The central concept in these chapters is that we exploit the explicit notion of visits to control and manipulate chunks of AG evaluation. It allows us to transform the tree during attribute evaluation—precisely what we need to express type inference algorithms. Many classic AG approaches use a notion of visits in their intermediate languages (Section 1.3.4). In this thesis, we instead propose to use visits as *programming model*. With this programming model, we express resolution strategies for declarative aspects of type rules.

Our extensions offer flexible ways to *tune* conventional attribute grammar evaluation, and are conservative extensions of (ordered) attribute grammars. We offer a delicate balance between on the one hand the implicit evaluation strategy of attribute grammars, and on the other hand the need to make this explicit for more complex evaluation strategies.

Underlying our extensions are well-defined concepts from higher-order AGs (first-class children), and ordered AGs (visits). Underlying these concepts are well-defined concepts from functional programming languages (first-class functions, coroutines, and referential transparency). These concepts form a solid theoretical basis to build upon.

The extensions that we present are not limited to type inference. In fact, type inference is a use case that sets challenges whose solutions improve the abstraction facilities that are available to structure compilers.

Thesis organization. Chapter 2 gives a detailed summary of the thesis. Subsequent chapters focus on individual extensions, and reintroduce relevant terminology. For background information related to type systems and attribute grammars, Section 1.2 and Section 1.3 can be used as reference.

Publications. The chapters of thesis are based on the following publications:

- We presented an earlier version of Chapter 3 at the Workshop on Generative Technologies (WGT '10) at ETAPS in 2010 [Middelkoop et al., 2010d]. An extended version appeared in the journal of Higher-Order Symbolic Computation [Middelkoop et al., 2011a].

- Chapter 5 is an extended version of the paper that we presented at the conference on Generative Programming and Component Engineering (GPCE '10) in 2010 [Middelkoop et al., 2010a].
- An earlier version of the chapter about GADTs (in the extended edition of this thesis) appeared in the post proceedings of Trends in Functional Programming (TFP '08) [Middelkoop et al., 2008], and a later version appeared in the journal of Higher-Order Symbolic Computation [Middelkoop et al., 2011b].
- Chapter 9 is to appear in the post proceedings of the symposium on Implementation and Application of Functional Languages.
- We presented Chapter 7 at the workshop on Language Descriptions Tools and Applications (LDTA '11) in 2011.

As a formal detail, the Association for Computing Machinery (ACM) has copyright on the paper version of Chapter 5 and Chapter 7. Elsevier has copyright on the paper version of Chapter 3.

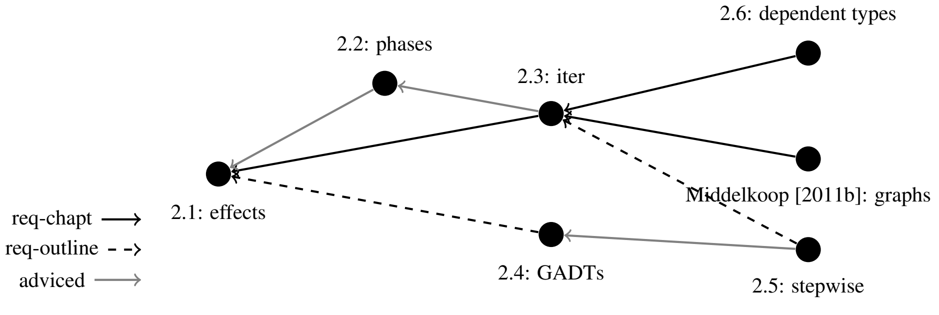
2 Outline of the RulerCore Concepts

Section 1.5 argued the necessity of extensions to attribute grammars. In the following chapters of this thesis we describe individual extensions to attribute grammars. In this chapter, we present the language RulerCore and give a detailed summary of the extensions. Each section summarizes a chapter in this thesis.

This chapter can be read before or after the other chapters. It shows how the individual chapters are connected together. This chapter uses a uniform notation, whereas in the individual chapters, we use minor differences in notation when that is more suited for that chapter. Consult the actual chapters for a more extensive explanation and technical material.

Outline. Chapter 3 and Chapter 5 give a detailed description of RulerCore's syntax. In this chapter, we use the syntax as described in Section 1.3.1. Prerequisite to this chapter are ordered attribute grammars (Section 1.3.4) and higher-order attribute grammars (Section 1.3.7).

The following dependency graph shows the dependencies between sections of this chapter (and the corresponding chapters). The solid arrows represent dependencies implied by the contents of the chapter, and the dashed arrows represent additional dependencies due to the presentation in this chapter. The dotted arrows represent a very weak dependency and lighter nodes are only present in the extended edition of this thesis:



2.1 Attribute Grammars with Side Effects

Ordered attribute grammars [Kastens, 1980] underly the extensions that we introduce in this chapter, and work out in the subsequent chapters of this thesis. In an OAG, attributes are evaluated in a fixed number of *visits* per node of the AST. Visits are a concept that play their role only in the evaluation algorithm of OAGs. In RulerCore, however, visits are a programming model: RulerCore has notation to specify visits so that the visits can be used to specify evaluation strategies.

In Chapter 3 we introduce the concept of a visit and their notation. This concept plays a central role, because it provides a model of the evaluation of ordered attribute grammars. We show that this model is powerful enough to get the effect of a visitor-pattern based traversals as known from the OO-world in terms of an attribute grammar based description, and we show how to deal with monadic or side-effectful operations. In this section we explain what a visit is, and introduce the notation for specifying these visits.

Visits and configurations. We first explain what a visit is. For that, we consider the evaluation of attribute grammars. An attribute grammar describes correctly decorated trees, but not how such a decoration is to be constructed. For OAGs, there exists an evaluation algorithm that starts with an initially undecorated tree and finishes with a correctly decorated tree. In this process, we pass though a sequence of intermediate states, with each intermediate state corresponding to a partially decorated tree.

Definition (State)**.** The *state* (or decorations) of a (partially) decorated tree consists of the local state of the root node of the tree and the states of its children. A local state of a node is a partial map from the attributes of the node to their values[1].

Definition (Defined attributes)**.** An attribute is *defined* when it is mentioned in the partial map.

Note that defined in this definition means that the attribute is part of the computed decorations of the tree. This definition is unrelated to rules defining attributes.

Definition (Configuration)**.** A *configuration* is a set of inherited and synthesized attributes of the root of a (partially) decorated tree, and describes which attributes of the root have associated values in the (intermediate) state of the tree. There exists some total order $\prec$ among configurations (we come back to this later). The total order $\prec$ must be stronger than the subset relation among configurations.

Thus, a configuration is an abstract description of an intermediate state.

Definition (Minimally defined state)**.** Given (static) dependencies between attributes, a tree is in a *minimally defined* state for a given configuration when precisely the attributes mentioned in the configuration have a value in the local state of the root and their (indirectly) dependent attributes have an associated value in the state of the tree.

Definition (Visit)**.** A *visit* is a state and configuration transition, which takes a tree[2] in a state as described by a configuration A to a tree in a state as described by a different configuration B with $A \prec B$.

[1] How the value of an attribute is represented depends on the implementation or the host language. Such a value can be an element in the domain of the attribute, but may also be a thunk. We usually assume that the values of attributes are at least in weak-head normal form (Section 1.2.2).

[2] A tree does not have to be the full tree, but may also be a subtree. The definition is not limited to the full tree. Indeed, a visit may require visits to subtrees: we usually describe visits per node of the tree, and specify what visits are performed to children of the node.

Thus, during a visit, computations are performed which determine the values of attributes. Usually, B contains at least one synthesized attribute that is not yet present in A, since that is a motivation to perform a visit[3]. Values for the inherited attributes in the set difference $B - A$ are provided by the parent[4] prior to the visit. The notation that we present below facilitates a statically finite and explicit description of state and configuration transitions induced by visits.

If we take a partial order among configurations instead of a total order, the order represents a Direct Acyclic Graph (DAG) where the vertices of the graph represent configurations and the edges represent visits. The evaluation of attributes for a tree associated to this DAG entails walking a path in the DAG where the parent chooses which visits to invoke on the tree. However, since we base our work on OAGs, we impose a total order that corresponds to a DAG with only a single path, which simplifies the implementation and the notation. In Section 2.6 we generalize our work so that the DAG is actually a tree.

Rationale. In this thesis, we distinguish two important notions of evaluation: a visit and a step. Visits (this section) provide a static model of evaluation, whereas steps (Section 2.5) provide a runtime model of evaluation. We argued in Section 1.5 that we wish to describe evaluation strategies, thus we do so in terms of the models as mentioned above. In this Section, we focus on the static model.

Every node in the tree is related to a production, and each production has an associated collection of rules. Attributes are computed by evaluating rules (Section 1.3.4). OAG evaluation is compositional in that it separates the evaluation of collections of rules of the parent from the evaluation of the children. In the case of evaluation of the parent, we make *internal visits* explicit, which statically describes the evaluation of a collections of rules of a node in the tree. In the case of the evaluation of children, we note that a visit to a child is the statically *smallest* unit of evaluation for a child that can be specified as part of the evaluation of the parent, and thus provides a model with a fine granularity.

Description of visits. RulerCore provides notation to describe visits. In a conventional AG, we declare attributes per nonterminal, and specify rules per production. In RulerCore, we additionally declare a linearly ordered sequence of visits and specify which attribute belongs to which visit. Also, we specify for each rule in which visit it is evaluated. Below, we describe the notation that forms an essential prerequisites for the remaining sections of this chapter. Chapter 3 provides extensive examples and technical background.

Definition (Interface declaration)**.** An *interface* declaration of a nonterminal specifies a linear sequence of visits to the nodes with which the nonterminal is associated.

Definition (Visit declaration)**.** Each visit declaration specifies which attributes must be defined prior to that visit, and which synthesized attributes become defined as a result of the

[3] In a conventional OAG, the *only* motivation for performing a visit is to get synthesized attributes computed, and this motivation is formalized as dependencies between attributes. Below, we show other motivations (e.g. to perform side effects) and show how these are formalized.

[4] We assume that each node has a parent. In case of the root node, the parent is represented by the interface with the host language (Section 1.3.1).

visit.

Configurations are not explicitly declared. The configuration of a node before a given visit is the union of the attributes declared for preceding visits, starting with the empty set. Visit declarations thus form a partitioning of the attributes of a nonterminal.

In the following example, for some *Expr* nonterminal, we declare a linear sequence of visits *analyze* and *compile*. We specify that in the first visit *analyze* a synthesized attribute *errors* is defined given the inherited attribute *env*. In the second visit *compile*, the synthesized attribute *output* gets defined, given the inherited attribute *optimize*, and the attributes defined earlier:

```
itf Expr   visit analyze   inh env      :: Env
                           syn errors   :: Errs
           visit compile   inh optimize :: Bool
                           syn code     :: Code
```

The order of appearance of visits matters, whereas the order of appearance of attribute declarations in a visit-block does not.

As we have seen, configurations are an abstract representation of the state of a node. A configuration records which attributes have been evaluated. Configurations are not explicitly named in the interface declaration. Instead, we associate with a visit the configuration that corresponds to the state at the beginning of the visit. So, during evaluation, the decoration of an *Expr* node is initially in the *analyze* state, then in the *compile* state, and finally in some final state.

The above example declares two visits for nodes associated with the *Expr* nonterminal. During evaluation, these visits correspond to a state transition. Such a state transition is described by a collection of rules which are specified per production using a semantics-block in conventional AG notation (Section 1.3.1):

```
sem Expr prod Var                         -- rules for production Var
  loc.defined = loc.x `member` lhs.env    -- tests whether ident loc.x is in the env
  lhs.code    = Code_Var loc.x            -- some translation to a target language
  lhs.errors  = if loc.defined then [ ] else [Undefined loc.x]
```

The above semantics-block introduces three rules for the production *Var*. Each rule is implicitly associated with a visit to *Expr*. Later we introduce notation to declare such a correspondence explicitly.

Default-notation. Before we continue with explicit notation for visits, we take a slight detour to introduce some notational conveniences. When programming with AGs, we often use copy rules (Section 1.3.12). Note that RulerCore is a *core language*, thus we prioritize implementation convenience over concise syntax. Copy rules are a typical front-end concept. However, copy rules may interact with RulerCore's evaluation algorithm, hence we model them explicitly.

RulerCore provides default-notation to improve on *copy rules*. For example, we can define the rules of production *App* with *default-rules*:

```
sem Expr prod App          -- rules of production App
  default env optimize     -- declares copy rules
  default errors = concat  -- declares collection rule for errors
```

In the example, the default-rules introduce generic rules for he inherited attributes *env* and *optimize*, and a collection rule for the synthesized attribute *errors*.

A default-rule specifies that the value of an attribute can be inferred from equally named attributes of the production. We provide several flavors of default rules for different generic situations, which each differ in how attributes are combined. The above rules are syntactic sugar[5] for rules of the following form:

```
r ::= ...               -- conventional rules (Section 1.3.1)
    | m o x̄ h           -- default rules for the attrs with name x̄
m ::= default0          -- applies even if no attrs matched
    | default1          -- at least one attr must match (default)
o ::= lexical           -- uses the lexical order of children (default)
    | lexicalrev        -- uses the lexical order in reverse
    | scheduled         -- determines order after scheduling
h ::= ε                 -- threaded behavior
    | = e               -- use expr e to combine a list of values of matching attrs
    | use e1 e2         -- applies the list algebra (e1, e2)
x                       -- attribute name
```

The order annotation *o* determines the order in which children are considered in the resolution process. In case of the scheduled-order annotation, the actual definition is determined after scheduling. As a notational convenience, a default-rule may be specified as part of the interface of a nonterminal, which then applies to all semantics-blocks of that nonterminal.

Explicit association to visits. In RulerCore, rules are associated with a visit. The association is by default implicit, but notation is available to specify the association explicitly by organizing the rules inside a visit-block:

```
sem Expr prod Var     -- semantics-block for production Var
  ...                 -- rules without an explicit association
  visit analyze       -- note the indentation (important in a later section)
    ...               -- rules associated with analyze (or later)
    visit compile     -- note the indentation (important in a later section)
      ...             -- rules associated with compile (or later)
```

With this notation, we specify constraints on the scheduling of rules, in addition to the conventional constraints imposed by value dependencies between attributes and rules. Via a cycle analysis, the constraints can be verified to be satisfiable using standard algorithms (Section 1.3.4). We come back to the rationale for the additional constraints later.

[5] We give the abstract syntax of the *desugared* rules, but use the sugared version in the code figures.

```
sem Expr prod Var                                       -- rules of production Var
  visit analyze                                         -- rules of the analyze visit or later
    loc.defined = loc.x `member` lhs.env
    lhs.errors  = if loc.defined then [ ] else [Err_Undefined loc.x]
    visit generate                                      -- rules of the generate visit
      lhs.code = Code_Var loc.x                         -- actually independent of any visit
sem Expr prod App                                       -- rules of production App
  default env optimize                                  -- declares copy rules
  f.env = lhs.env                                       -- explicitly written rule
  visit generate                                        -- rules of the generate visit
    a.optimize = lhs.optimize                           -- explicitly written rule
    lhs.errors  = concat [f.errors, a.errors]           -- explicitly written rule
```

Figure 2.1: Examples of organizing rules in a visit-block.

Definition (Visit semantics)**.** A visit-block t is explicitly associated to some visit x and may contain rules and a nested visit-block. The rules may be evaluated during visit x or a later visit.

The following is the grammar of a semantics and visit-block:

```
s ::= sem N prod P $\overline{r}$ t    -- common rules r and visit blocks t
t ::= visit x $\overline{r}$ t         -- common rules r and subsequent visit t
   |  $\varepsilon$                    -- terminator of sequence of visit blocks (implicit)
```

A visit-block is associated with a similarly named visit declared on the interface of the nonterminal. However, not every visit is necessarily associated with a visit-block. The same name may not occur twice, and the total order must be preserved: If x is the name of a visit-block and x' the name of a nested visit-block then $x \prec x'$.

Figure 2.1 gives an alternative way to organize the rules of the earlier example. The rules inside a visit-block may appear in any order without affecting the semantics of the grammar.

A visit-block introduces a scope for local attributes such as *loc.defined*. When defined in a visit, such an attribute is only visible inside the visit-block it is defined in, and its enclosed visit-block. The scoping plays a role in Chapter 5 where rules may additionally be organized in clauses. The inherited and synthesized attributes of the children and *lhs* are globally scoped per production.

Exploiting visits: invoke rules. The additional refinements on the scheduling of rules can be used to allow side-effects in our specification (motivated later). For this purpose, we introduce two additional forms of rules.

Definition (Invoke rule)**.** An invoke rule specifies properties of a visit to a child. These properties usually specify some evaluation strategy.

The following is the grammar of invoke-rules of which we explain some properties below:

$r ::= \ldots$	
$\mid$ **invoke** x **of** $\overline{c}\ z$	-- specifies visit x of children $\overline{c}$ with strategy z
$z ::=$ **implicit**	-- determined only by attribute dependencies
$\mid$ **explicit**	-- invoke-rule restricted to the visit it appears in
$\mid$ **parent**	-- invoke-rule restricted to visit with the same name

The strategies are explained later. Invoke-rules are optional. When a visit x to a child c is not explicitly specified, it is implicitly specified as the rule:

invoke x **of** c **implicit**

The invoke-rule is annotated with a strategy, which provide a means to specify properties of the evaluation of a child. The above strategies constrain when visits to children can be performed. We will later see more strategies.

The implicit-strategy (above) specifies no additional constraints. The explicit-strategy (above) requires the invoke-rule to be nested in a visit-block, and constrains the visit to the child to the evaluation of that visit-block. The parent-strategy requires that the parent has an equally named visit and constrains the visit to the child to that visit of the parent. An empty list of children in the invoke-rule applies the strategy to all children which have a visit defined with the same name.

Exploiting visits: rules with side effects. We allow rules with side effects, but restrict these to the introduction of children only.

Definition (Side-effectful child-rule)**.** A *side-effectful child-rule* is a child defined by some impure expression.

A child-rule must appear in a visit-block, and its application is restricted to that visit. The syntax of child-rules is (similar to Section 1.3.7):

$r ::= \ldots$	
$\mid$ **child** $c : N \leftarrow f\ [\overline{a}]$	-- definition of child with side effects

The scheduling guarantee is that the impure expression f is applied before the end of the visit. The relative order of the side effects within a visit is however not specified[6]. This coarse-grained way of specifying the evaluation order allows us to safely integrate side-effectful operations in the attribute grammar, while not having to micro-manage the order of the side effects.

In case of Haskell as target language, f is a monadic expression that yields the structure of child c given values of attributes $\overline{a}$. For example, along the lines of Section 1.3.12, we can define such children to encode operations that provide fresh type variables and perform unification. Instead of threading a substitution, we pass these children an inherited *IORef* to

[6] We actually provide a notion of *internal visits* to specify the relative order of side effect within a visit.

```
sem_App s_f s_a = lhs_analyze where            -- body of the function
  lhs_analyze lhs_env = do                     -- body of the first visit
    f_analyze ← return s_f                     -- monadic child rule
    a_analyze ← return s_a                     -- monadic child rule
    let f_env = lhs_env
        a_env = lhs_env
    (f_errs, f_compile) ← f_analyze f_env      -- invoke rule
    (a_errs, a_compile) ← a_analyze a_env      -- invoke rule
    let lhs_errs = concat [f_errs, a_errs]
        lhs_compile lhs_optimize = do ...      -- body of the second visit
    return (lhs_errs, lhs_compile)             -- results of first visit
```

Figure 2.2: A sketch of the coroutine *lhs_analyze*.

a substitution. This allows us to schedule the effects of unification in a less strict way than the explicit threading of substitutions would entail[7].

Scheduling and coroutines. In the context of this chapter, we shall refer to augmented production dependency graphs as PDGs. Section 1.3.2 explains how dependency graphs are obtained from AG descriptions. In this section, I/O graphs are Nonterminal Dependency Graphs (NDGs) because of the explicitly declared single visit sequence per nonterminal. From the PDGs, an execution plan with an as-late-as-possible scheduling of the rules can be obtained if the PDGs are cycle-free. Since we introduced new rules, the question arises how these rules affect the evaluation of the grammar. Also, RulerCore's NDGs leave less freedom for scheduling the rules in the PDGs, since we need to adhere to the explicitly defined interfaces.

We introduce the language RulerBack as a desugared variant of RulerCore which represents execution plans[8]. In comparison to RulerCore, in a RulerBack all implicit syntax is made explicit and rules in a RulerBack description are totally ordered. In this thesis, we define mappings of RulerBack to algorithms in various host languages.

Section 1.3.5 shows that ordered AGs can be implemented with coroutines. The example below serves as a sketch of a mapping from RulerBack to *monadic* coroutines. For each production P, we introduce a semantic function *sem_P*, which takes the coroutines *s_f* and *s_a* as parameter that serve as children f and a and produces a coroutine *lhs_analyze* for the first visit *analyze*. The coroutine for some visit x is a function *lhs_x* that takes the values of the inherited attributes of visit x, and produces a monadic tuple with values of the synthesized attributes of visit x and a coroutine for the successor of x. The coroutine *lhs_optimize* (Fig-

[7] In Chapter 5 we show how to treat nodes as first class value, and show in Middelkoop [2011b] some complex traversal patterns that do not follow the tree structure. In such situations, we need more flexibility in the scheduling of side-effectful operations.

[8] RulerCore can be considered a programming language for execution plans of AGs.

ure 2.2) is thus constructed as part of the body of *lhs_analyze*. An attribute $k.x$ is transcoded as the Haskell identifier *k_x*. The monad serves as an abstraction for evaluation algorithms. We exploit the monadic structure in later chapters.

Foundation. The fragment of RulerCore that we introduced so far does not add to the expressive power of attribute grammars: it can be expressed as a conventional attribute grammar, which we do so below to be more precise about the semantics of the notation. We assume that the RulerCore description is desugared to RulerBack, which we mentioned earlier. Below, we describe how to map the RulerBack description to a conventional attribute grammar. In general, interface declarations are be mapped to attribute declarations by erasing visits. Semantics-blocks in RulerCore are translated to semantics-blocks by erasing visit-blocks and invoke-rules.

To represent the erased information in a conventional attribute grammar, we thread an additional attribute through the tree for each visit. These attributes serve as synchronization points for the beginning and end of the visit. Per visit v of a nonterminal N, we introduce two additional attributes $begin_v$ and end_v:

$$\begin{array}{ll} \textbf{attr}\ N & \textbf{inh}\ begin_v :: S\ T \\ & \textbf{syn}\ begin_v :: S\ T \end{array}$$

The attributes model the side effects by encapsulating a state as some type $S\ T$, which we come back to later.

Also, we thread these attributes *through the rules* to enforce their evaluation order. Per production, we associate a unique consecutive number (starting from 0) with each rule in a production. This is possible because there exists a total order among the rules. For each rule, we introduce a local attributes $loc.begin_i$ and $loc.end_i$ where i is the number associated with the rule. The purpose of the attributes is to mark respectively the beginning and end of the evaluation of the rule. We show later how to make a rule dependent on its *begin* attribute (in addition to its normal dependencies), and how to make the *end* attribute dependent on the evaluation of the rule.

For a visit v and production P of nonterminal N there exists a collection R of rules that are associated with v after scheduling. If this collection is not empty, let k be the lowest number associated with the rules, and l the highest number. We then connect the begin and end of the visit with the beginning and end of the rules associated with the visit:

$$\begin{array}{lll} \textbf{sem}\ N\quad \textbf{prod}\ P & & \\ \quad lhs.begin_v = lhs.end_j & & \text{-- if } R \text{ is empty} \\ \quad loc.begin_k = lhs.begin_v & & \text{-- otherwise } R \text{ not empty} \\ \quad lhs.end_v\ \ = loc.end_l & & \text{-- otherwise } R \text{ not empty} \end{array}$$

When a rule with associated number i and a rule with number $i+1$ are in R, then we add also:

$$\begin{array}{l} \textbf{sem}\ N\quad \textbf{prod}\ P \\ \quad loc.begin_{i+1} = loc.end_i \end{array}$$

```
class ThrEff t where
  type M t :: * → *    -- type of a monadic computation
  type S t :: *        -- type of the state
  impure :: M t α → S t → (S t, α)
  pure   :: α     → S t → (S t, α)
  pure (x, s) = s `seq` x `seq` (s, x)
```

Figure 2.3: API of threaded effects.

```
data IsPure    -- do not thread a state
instance ThrEff IsPure where
  type M IsPure t  = t
  type S IsPure    = ()
  impure           = pure
data IsIO      -- thread state of the world
instance ThrEff IsIO where
  type M IsIO      = IO
  type S IsIO      = State# RealWorld
  impure (IO m) w = case m w of (# w', a #) → a `seq` (w', a)
```

Figure 2.4: Example instances for threaded effects.

At this point, we chained the attributes together, except that still attributes $loc.begin_i$ needs to be connected to $loc.end_i$, and the begin and end of visits to the children need to be incorporated.

To thread the attribute through a rule, we introduce in Figure 2.3 the functions *pure* and *impure* which depend on the type T. The function *impure* takes an effectful computation $M\ t\ \alpha$ and an initial state $S\ t$, then produces an updated state paired with the result of the computation. The function *pure* passes the state on unchanged. Figure 2.4 gives some exemplary instances. An instance of *ThrEff* can be given for any monad that threads a state.

To complete the chain, we show the mapping of rules of RulerCore. The above functions are used in the translation of a rule r with associated number i to $[\![\, r \,]\!]_i$:

$$\begin{array}{lcll}
[\![\, \textbf{invoke}\ v\ \textbf{of}\ c \,]\!]_i & \leadsto & c.begin_v & = loc.begin_i \\
 & & loc.end_i & = loc.end_v \\
[\![\, \textbf{child}\ c : N \leftarrow f\ [\overline{a}] \,]\!]_i & \leadsto & (loc.end_i, loc.c) & = impure\ (f\ [\overline{a}])\ loc.begin_i \\
 & & \textbf{child}\ c : N & = loc.c \\
[\![\, p\ [\overline{a_2}] = f\ [\overline{a_1}] \,]\!]_i & \leadsto & (loc.end_i, p\ [\overline{a_2}]) & = pure\ (f\ [\overline{a_1}])\ loc.begin_i
\end{array}$$

The concept of visits thus provides a means to reason about attribute grammars with side-effectful computations in their rules. Note that although the AG description may be thought of as having side effects, the underlying model is still purely functional.

Remarks. We purposefully allow side effects only in the creation of children. Conventional rules must be purely functional. This ensures that the attributes have a referentially transparent definition, even though the tree structure itself not[9]. Chapter 7 shows how to implement search algorithms, and introduces syntax to define children without inherited attributes. For advanced search algorithms, which make use of sharing and memoization, limited use of side effects plays an important role.

More generally, by making visits explicit, we can integrate our approach with compilers that are built on top of monads, and to use tree traversals in impure environments. Also, we can use the side effects to efficiently access results from nodes visited earlier in a traversal. This can be used to implement memoization strategies.

The visitor pattern [Gamma et al., 1993] is often employed to implement recursive traversals over tree structures in imperative languages. Concretely, Chapter 3 presents how our approach generalizes over the visitor design pattern. For this purpose, we use JavaScript as a host language, which in passing shows that our extensions are applicable to domains other than the implementation of type inference algorithms or functional programming languages. In this context, a *visitor* is an object that contains an algorithm that describes which children to visit, and what changes to apply to the state of the node, or the visitor itself. Attributes provide a convenient way to access the state of nodes through attributes, and with our approach the changes to the state of the visitor can be encoded with side effects. With respect to these attributes, our approach offers the benefits of AGs, including the static enforcement that attributes are defined before they are used.

2.2 Attribute Grammars with Commuting Rules

In Chapter 4, we generalize visits to a *phases*. A visit is a technical more internal concept which precisely controls the evaluation of the grammar. A phase is a more abstract concept which the programmer can use to specify properties of the evaluation of the grammar. To reason with side effects in this setting, we present *commuting* rules, which are rules with relaxed dependencies.

Phases. We start with the notion of a phase:

Definition (Phase)**.** A *phase* represents a sequence of state transitions, controlled and observable by the parent, which take the node's state to a state described by its next configuration.

Such a state transition consists of a sequence of smaller state transitions, which correspond to the visits as described in Chapter 3. A nonterminal may be associated with a set of possible visit sequences for its phases, and a production specifies for each of its children which sequence to take.

[9] We can encode any rule as an attribute-defined child (Section 1.3.7), thus the restriction does not limit expressiveness. The purpose of the restriction is to ensure that side effects are sufficiently contained.

```
itf Block                 -- phase interface (visit-interfaces are not given anymore)
  phase analyze           -- analyze phase
    inh pred  :: Lab      -- label to be used as predecessor from the predecessor
    inh succ  :: Lab      -- label to be used as successor from the successor
    syn pred  :: Lab      -- label to be used as predecessor for the successor
    syn succ  :: Lab      -- label to be used as successor for the predecessor
  phase transform         -- transformation phase
    inh debug :: Bool
    syn trans :: SELF     -- self attribute (Section 1.3.6)
    default debug         -- default rule on interface for inh attr
```

Figure 2.5: Phase interface of a *Block*.

We illustrate the above with an example. Suppose that we describe an analysis and transformation of a tree of labeled instruction blocks. The abstract syntax of blocks is described by the following grammar:

```
grammar Block  prod Seq   nonterm l, r   : Block
               prod Leaf  term    lab   :: Lab
                          term    instr :: Instr
```

The actual transformation of the instructions is out of the scope of this example. Let *transform* be a function that requires the label of the predecessor and sucessor to transform the instructions in the leafs.

To apply the *transform* function, we associate the label of a preceding and succeeding block with each instruction. The chained attribute *pred* represents the label of the left-nearest instruction, and attribute *succ* the label of the right-nearest instruction. Effectively, we pass *pred* from left to right, and *succ* from right to left. The *phase interface* for a nonterminal declares these attributes is given in Figure 2.5. Attributes may be declared outside phases. The ordering of phases is deduced from semantics blocks, thus not from the order of appearance in the phase-interface specification: phases represent non-overlapping units of evaluation.

As part of the semantics for productions of *Block*, we describe the flow of the *pred* and *succ* attributes in Figure 2.6. Seq-productions act as crossbar switches, and Leaf-productions inject their labels in the attribute flows. Since we have inherited and synthesized attributes with the same name, we use the prefixes *inh* and *syn* to explicitly distinguish these attributes. Assume that *Block* is also the root symbol, for which we at the root provide initial values for the inherited attributes, and request values for all synthesized attributes.

Implementation of phases. Since a phase effectively represents a unit of evaluation, we can choose an algorithm for the evaluation of a phase. We assume here that we choose a statically ordered evaluation algorithm, which reduces the choice to either a Kastens style or a Kennedy-Warren style algorithm (Section 1.3.5).

```
sem Block prod Seq     -- rules related to pred and succ
   inh.l.pred    = inh.lhs.pred    -- left to right
   inh.r.pred    = syn.l.pred
   syn.lhs.pred  = syn.r.pred

   inh.r.succ    = inh.lhs.succ    -- right to left
   inh.l.succ    = syn.r.succ
   syn.lhs.succ  = syn.l.succ
sem Block prod Leaf
   syn.lhs.pred  = loc.lab         -- is predecessor of next
   syn.lhs.succ  = loc.lab         -- is successor of prev

   loc.newInstr  = transform loc.instr inh.lhs.pred inh.lhs.succ lhs.debug
   syn.lhs.trans = Leaf loc.lab loc.newInstr
```

Figure 2.6: The semantics of productions of *Block*.

A Kastens-style algorithm does not suite the example. In the example, the rules for *pred* and *succ* are independent. However, the attributes of the analyze-phase need to be computed in at least two visits. The rules for production *Seq* require either *succ* or *pred* to be computed first. This is a typical example where a Kastens-style scheduling [Kastens, 1980] fails to find an ordering, because that scheduling induces extra edges in the PDG, which for this example causes a cycle.

Kennedy and Warren [1976] describe an algoritm that does not induce extra edges in the PDG. A set of visit sequences is determined for each nonterminal, such that there is one visit sequence per context of an occurrence of the nonterminal symbol. We present a variation on this algorithm that schedules rules as late as possible, and only those that needed in a given context. Moreover, we show how to represent such visit sequences in a strongly-typed functional language.

Visits-DAG. We associate a graph structure with each nonterminal which represents the visits of that nonterminal:

Definition (Visits-DAG)**.** A *visit-interface DAG* describes the set of visit interfaces that are associated with a nonterminal. The graph has exactly one source vertex, and at least one sink vertex. Each vertex represents a configuration, each arrow a visit, and each path from the source to some vertex a visit-interface. To disambiguate, we may call a vertex in the visits-DAG a *visits-vertex*.

For the above example, Figure 2.7 shows the visits-DAG. There are at least three paths in the visits-DAG. The middle path represents the root where all inherited attributes are available, and two other paths represent the respective first knowledge of one of the inherited attributes.

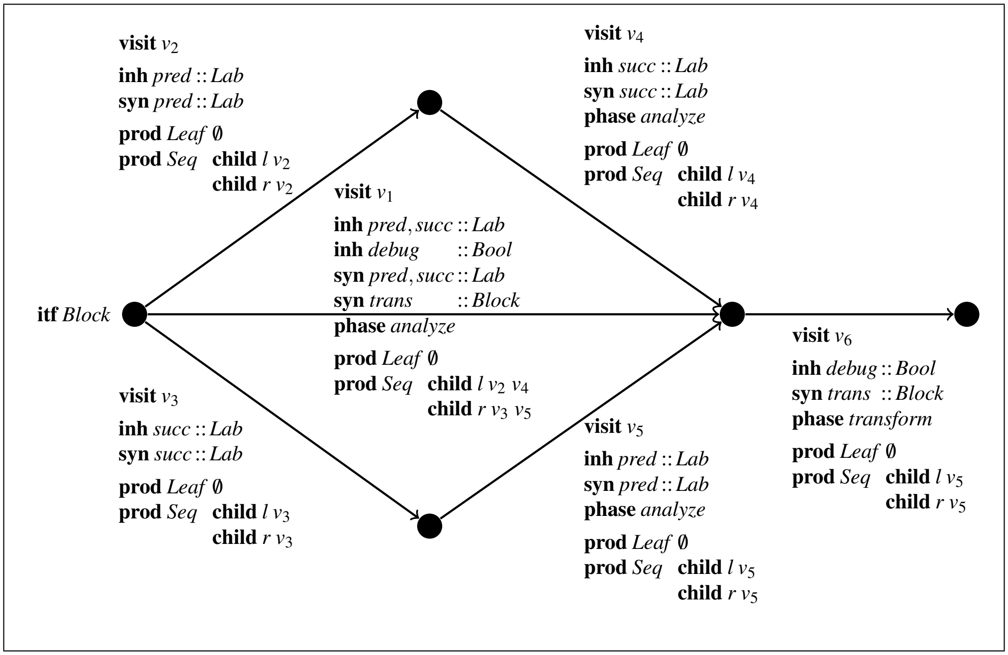


Figure 2.7: Visits-DAG of the example.

Each edge has at least one *output*, which is either a synthesized attribute or phase ending. Along each path, the number of attributes increases. Each edge corresponds with a visit; we gave each a unique label v_i. Also, we associated with each edge some meta-data regarding productions: the visits performed on the children of the productions during the execution of visit that is associated with the edge.

Paths may be of different length, and end in different configurations. In the above example, all paths end in the same configuration because in each context all attributes are eventually needed.

Section 4.5 describes the visits graph and its properties in more detail and shows how to incrementally compute it. Given this graph, for each edge and each production, a collection of RulerBack rules can be determined. The branching-factor of each node determines code duplication. In practice, this code duplication is not a reason for concern. The visits graph of the largest AG of UHC has about 10,000 edges and already leads to a tractable implementation. With some optimizations (Section 4.6), we reduced this number to about 3,000 edges.

Commuting rules. Section 2.1 shows a translation of visits to conventional attribute grammars using a functional encoding of the state. The translation involved adding an additional chained attribute (the state attribute) per visit which represents the state and a transformation of the rules to thread the attribute through the rules scheduled for the visit. A property of this approach is that the side effects that arise from visit a child of a parent can only be observed by the parent or its other children by inspecting the state attribute of the child after the visit.

A similar translation is possible for phases. Analogously, a parent only observes the side effects arising from a child after completing the phase. However, during the evaluation of a phase of a child of a parent, side-effectful rules of the parent or the other children of the parent may be evaluated, since a phase consists possibly of multiple visits to a child. In this situation, with a single chained attribute per phase, side effects arising from a child may not be timely observed in the parent or in siblings, and such a translation does not fully capture the semantics of side-effectful rules.

A possible approach is to translate the phases to an explicit visit sequence, and then use the translation of Section 2.1. However, visits are implicit in the phases model and additionally there may be a visit sequence per context. Instead, we take the opportunity to present *commuting rules*.

Definition (Commutative compositions and commutable rules)**.** Given an explicit ordering of rules, the composition of two rules is *commutative* when the two rules are *commutable*, which means that the rules may be swapped in the composition without affecting the intended result.

Rule composition is a conditionally commutative operator. Commutable rules can be considered as commutable operations. The swapping of rules models side effects, and commutativity permits reasoning about the safe use of side effects.

A semantic tree is a composition of the rules of the tree (Section 1.3 and Section 1.3.4). Section 1.3.9 shows that a composition of rules can be expressed with arrow notation, which is a convenient notation to define when two rules are commutable. Further, we wish to refer describe a composition of rules (e.g. the composition of rules of a node) in a larger context (e.g. the composition of rules in a tree). We define a *rule context* h as a composition of rules with a hole in it so that $h\ c$ represents the composition of rules with c the composition of rules at the location of the hole.

Definition (Commuting rule)**.** A *commuting rule* is a rule of the form $(x', y') = f\ x\ y,\ x \diamond x'$ where the letters x, y, etc. are (sets of) attribute occurrences and f is a semantic function.

A commuting rule thus only differs in the notation $x \diamond x'$, which denotes that the rule may be swapped with rules that define x or use x' (with renaming of the attribute occurrences). Such a rule is said to commute over x and x'.

Consider a rule r_1 of the form $(x_1, y_1) = f\ (x_0, y_0),\ x_0 \diamond x_1$ and a rule r_2 of the form $(x_2, y_2) = g\ (x_1, y_1),\ x_1 \diamond x_2$. In a composition of rules containing r_1 and r_2, the additional notation specifies that r_1 may appear ordered before r_2 and vice versa. Without the additional notation the rule r_1 must appear before r_2 because r_2 refers to an attribute defined by r_1.

With commuting rules as AG feature, side effects can be encoded as a single chained attribute per nonterminal threaded through each rule and having each rule commute over this attribute. Such a translation is more straightforward than the translation in Section 2.1 and also works for phases.

Referential transparency. At the level of specification, the use of commutable rules may break referential transparency and thus complicate equality reasoning. A question that arises is how to reason with a safe use of commutable rules. We define below a law for this purpose.

Suppose that $r_1 \not\prec r_2$ denotes that r_1 is independent of r_2 with respect to the dependencies between attributes and rules except for the dependencies between attributes where the rules commute over. There are two compositions in arrow notation (c_1 and c_2) to consider:

```
c₁ = proc (x₀,y₀,z₀) → do        -- composition of r₁ and r₂
        (x₁,y₁) ← f ≺ (x₀,y₀)    -- rule r₁ in arrow notation
        (x₂,z₁) ← g ≺ (x₁,z₀)    -- rule r₂ in arrow notation
        returnA (x₂,y₁,z₁)
c₂ = proc (x₀,y₀,z₀) → do        -- composition of r₂ and r₁
        (x₁,z₁) ← g ≺ (x₀,z₀)    -- rule r₂ in arrow notation
        (x₂,y₁) ← f ≺ (x₁,y₀)    -- rule r₁ in arrow notation
        returnA (x₂,y₁,z₁)
```

The identifiers y_0 and z_0 represent the independent input attributes of respectively r_1 and r_2, and identifiers y_1 and z_1 their respective independent output attributes. Identifiers x_0, x_1 and x_2 represent the attributes over which the rules may commute.

Definition. Rule context A rule context h is a function that serves as an abstraction of a rule composition with a hole. It takes as parameter the composition to fill to hole with.

Definition (Commutable over attributes)**.** We now say that r_1 and r_2 are *commutable* over attributes of x_0,x_1 and x_1,x_2 if $r_1 \not\prec r_2$ if their compositions c_1 and c_2 give an equivalent results $h\ c_1 = h\ c_2$ in some given rule context h.

When the rules are commutable, the outcome of swapping the rules in rule context h is equivalent, thus the slide of the grammar that contains the rules r_1, r_2 and those in h is referentially transparent. Rule context h should be chosen in such a way that it expresses how the context of the rules interprets the attributes computed by the rules.

We finish this discussion of commutable rules below with an exemplary definition of h which states that for an attribute that provides fresh numbers only uniqueness is relevant.

For some exemplary grammar of expressions, the following two rules thread a counter k, and extract two unique numbers in $loc.u_1$ and $loc.u_2$:

```
sem Expr prod Var
  (loc.k, loc.u₁)      = f (inh.lhs.k, inh.lhs.k), inh.lhs.k ◇ loc.k    -- rule one
  (syn.lhs.k, loc.u₂) = g (loc.k + 1, loc.k),     loc.k ◇ syn.lhs.k    -- rule two
```

With the following definitions for f and g:

$$f\ (a, \_) = (a+1, a)$$
$$g\ (b, \_) = (b+1, b)$$

For the following definition of h, the above two rules are commutable. In both compositions of the rules, the two resulting numbers are different from each other:

```
h r n = a ≢ b where         -- abstraction: the unique numbers should be distinct
  (_, a, b) = r (n, (), ())  -- for any number n that represents the inh.lhs.g
```

We may choose functions h that state stronger invariants and take more context into account. For example, when we consider the collection of a list of error messages, we may take the slice of the rules that depend on the error messages, and require that the lists are equal when ordered according to the source location of each message.

Commutable rules can be applied when expressing collection attributes as a chained attribute. A collection attribute is a synthesized attribute with a commutative monoid or trace monoid as value. Often, such attributes can be encoded more efficiently as a chained attribute. However, threading a chained attribute through some children may induce tighter dependencies than combining synthesized attributes of these children, and thus reduces freedom in attribute scheduling and may even lead to cycles. With commutable rules a chained attribute can be used without the tighter dependencies.

Remarks. In Chapter 4 we work out phases in more detail. In this chapter, we describe how to compute the dependency graphs and how to perform scheduling of phases. This work shows how to generalize visits to phases, and allows us to describe the extensions that we present in the next sections (and their corresponding chapters) using visits, so that we factor out the dependency graphs and scheduling in the next sections and chapters.

2.3 AGs with Tree Construction

In Chapter 5, we show several AG extensions based on the model of explicit visits that allow us to conditionally and iteratively define attributes and children. Additionally, we use annotations on visits and invocations of visits to fine-tune evaluation strategies.

In this section, we give an overview of the extensions. In a conventional attribute grammar the rules to evaluate for a node are the rules associated to the production that is associated to that node. As extension, we wish to have more fine-grained control over which sequence of rules is evaluated. Therefore, we split up visit-blocks in a sequence of clause blocks. Each clause-block may contain rules, and per visit some strategy choses which clause-block to use to compute the attributes. In this section, we take a fixed strategy based on the order of appearance of clause-blocks and backtracking. We show that with this approach we can implement a dispatch of rules based on values of inherited attributes. In Section 2.5 we present a mechanism based on a stepwise evaluation to actually define custom strategies.

Further, we show how to express iteration by annotating invoke-rules with a strategy that repeats the evaluation of the visit until a condition is met. Moreover, first-class children are an extension that allow children to be detached as value, or attached from a value. The techniques combined provide a powerful mechanism to encode fixpoint computations.

In the remainder of this section, we explain these extensions one-by-one. In Chapter 5 we show how these extensions are implemented.

Clauses. Instead of associating a collection of rules per production, we organize the rules in a different way. We associate a DAG with a production. Each vertex is associated with a configuration and each edge is associated with a visit and with a collection of rules. There may be multiple vertices associated to the same configuration, although there may only be

```
itf Expr visit check inh env    :: Env    -- environment containing declared types
                     inh tp     :: Ty     -- expected type of the expr
                     syn errors :: Errs   -- result of type checking
sem Expr prod Var visit check             -- cases for the check-visit of the var-prod
  clause defIdent                         -- case for when the variable is in the env
    match (Just loc.declTp) = lookup loc.nm lhs.env
    internal matchTp                      -- internal case distinction
      clause typeOk                       -- case for when the type matches
        lhs.errors   = [ ]
        match True = lhs.tp `isInstance` loc.declTp
      clause typeFail                     -- case in case of a type mismatch
        lhs.errors   = [Mismatch lhs.tp loc.declTp]
  clause undefIdent                       -- case for when the variable is undefined
    lhs.errors = [UndefVar loc.nm]        -- assumes a match of defIdent failed
```

Figure 2.8: Example of clauses.

one source vertex, wich must be associated with the empty configuration. During evaluation, a path is traversed through the DAG: one edge per visit and a strategy associated with vertices dictate which edge to traverse.

Definition (Clause)**.** A *clause* is an edge in the DAG as specified above.

As notational simplification, we impose the restriction that the DAG[10] must be a tree, and present notation below on how to describe this tree. Essentially, the rules are organized in clause-blocks per visit-block.

We start with an example in Figure 2.8 before explaining the notation. The example consists of a type checker for some var-production of a lambda calculus. In the example, we use clauses to encode case distinction. We distinguish a clause *defIdent* for when the identifier is in the environment, and a clause *undefIdent* when this is not the case. Moreover, we split the *defIdent* clause in two more clauses depending on whether the expected type matches the declared type using an internal visit.

The nesting of clauses forms a decision tree. A path in this tree is the sequence of clauses that are selected to compute the outputs of the visit. For now, we assume that clauses are selected with a fixed strategy based on the order of appearance (to which we come back later). Internal visits can be considered as ε-edges in the DAG as mentioned above.

The following changes to notation allow visits to consist of a non-empty ordered sequence of clauses:

[10] We thus identify two important DAGs: a DAG per nonterminal which describes visits and attributes, and a DAG per production which describes different sets of rules for visits to compute the attributes. The restrictions that we impose on the DAGs simplify the implementation or the notation.

$$\begin{array}{lll}
t ::= & \textbf{visit}\ x \quad \overline{r}\,\overline{k} & \text{-- conventional visit block, common rules } r\text{, and alternatives } \overline{k} \\
\mid & \textbf{internal}\ x\ \overline{r}\,\overline{k} & \text{-- internal visit block, common rules } r\text{, and alternatives } \overline{k} \\
\mid & \varepsilon & \text{-- terminator of visit/clause branch (optional)} \\
k ::= & \textbf{clause}\ x\ \overline{r}\ t & \text{-- clause-block with rules } \overline{r} \text{ and visit } t \\
x & & \text{-- identifier of a visit or clause}
\end{array}$$

Each clause contains a set of rules. This set of rules defines the synthesized attributes of the visit, and potentially subsequent visits. Thus, each clause provides a number of alternative definitions of the synthesized attributes of a visit.

We distinguish conventional visits and *internal visits*. A conventional visit is invoked by the parent and declared as part of the interface of the nonterminal. Internal visits and clauses are evaluated as part of the evaluation of their encapsulating visit or clause.

To describe the clause selection strategy, we distinguish three types of outcome for rules, visits, and clauses. Evaluation either *succeeds* with resulting attribute bindings, *terminates exceptionally*, or *fails* with a recoverable failure:

- Clauses are evaluated in the order of their appearance. The first clause that succeeds or terminates is chosen as the clause that provides the outcome of the visit. If a clause fails, the next clause is evaluated.
- During the evaluation of a clause, the rules are evaluated in a scheduled order. When a rule succeeds, the next rule is evaluated. If all rules succeed, the clause succeeds. However, a clause fails if the evaluation of a rule terminates exceptionally or fails.
- A visit terminates if any of its evaluated clauses terminate, and otherwise succeeds if a clause succeeds. If all of its clause fail then the visit terminates exceptionally or fails recoverable. The respective difference is made by whether the visit is annotated as *total* or annotated with a *partial* strategy. Visits are declared as total by default.

There are two types of rules that may fail:

- An invoke-rule may be annotated with a *partial* strategy. If it is, the invoke-rule fails if the visit to the child fails. Otherwise the invoke-rule either succeeds or terminates exceptionally.
- We present *match-rules* to specify conditions. A rule **match** $p = e$ requires that the value of e satisfies the pattern p, otherwise evaluation for the match-rule fails.

In the rule ordering, match-rules must be evaluated as part of the clause in which it is declared. Moreover, match-rules take priority in the rule scheduling. If two match rules are independent of each other, then the order of appearance determines which rule is scheduled first. As notational convention, we usually write match-rules up front in code examples.

Chapter 3 describes the evaluation of clauses as a generalization of productions. Chapter 5 describes an implementation in Haskell. Also, Chapter 7 shows how to evaluate clauses simultaneously.

In comparison to Conditional Attribute Grammars [Boyland, 1996] or conditionally defined rules in general, clauses allow us to define a condition for multiple attributes and also children.

```
grammar Eq prod    Check                          -- multiple clauses below
itf Eq visit check                                -- checks type equality
  inh tp1     :: Type
  inh tp2     :: Type
  syn errs    :: Errs                             -- outcome of check
sem Eq prod Check visit check                     -- semantics of Check prod
  default errs = concat                           -- collect type errors
  clause twoInts                                  -- when tp1 and tp2 are Ints
    match Ty_Int               = lhs.tp1
    match Ty_Int               = lhs.tp2
  clause twoArrs                                  -- tp1 and tp2 both arrow-types
    match (Ty_Arr loc.a loc.b) = lhs.tp1          -- tests if lhs.tp1 is an arr
    match (Ty_Arr loc.c loc.d) = lhs.tp2          -- tests if lhs.tp2 is an arr
    child u1 : Eq              = sem_Check        -- recursion on both arg-types
    child u2 : Eq              = sem_Check        -- recursion on both res-types
    u1.tp1   = loc.a;   u1.tp2   = loc.c          -- definitions of inh attrs of u1
    u2.tp1   = loc.b;   u2.tp2   = loc.d          -- definitions of inh attrs of u2
  clause mismatch                                 -- catch-all clause
    lhs.errs = [Err_Mismatch lhs.tp1 lhs.tp2]     -- error for each mismatch
```

Figure 2.9: Matching example.

Multi-attribute dispatch. Clauses provide a convenient way to describe the structure of a derivation tree when the structure of the tree depends on the values of attributes. For example, to prove type equality, the structure of the derivation tree is a determined by two attributes that represent the types in question.

The example in Figure 2.9 shows a first-order matching algorithm for the construction of a derivation tree for an equality judgment. Given two attributes tp_1 and tp_2 which values represent types (in some object language), we match *pointwise* against the structure of these types. The value of such an attribute is either the integer type constructor or a function type constructor. During evaluation, the derivation tree is constructed up to the points that the types match. The result of evaluation is an attribute *errs* that contains an error message for each type mismatch. The production *Check* does not declare any terminal nor nonterminal symbols. The example relies on higher-order children and clauses instead.

Iteration. A judgment $R\,\overline{p}$ can be seen as a constraint R between parameters $\overline{p}$ where R is a relation. Fixpoint iteration is often employed to gradually construct a solution to a set of such constraints.

In Figure 2.10 we show how to encode fixpoint iteration in AGs by iterating visits. We use some extensions of previous sections and Section 1.3.12 to keep the description concise. We

```
grammar Top prod Top nonterm root : Constrs     -- root symbol
grammar Constrs = [Constr]                      -- short hand for cons-list
grammar Constr prod Subset term a, b :: Ident   -- subset constraint

itf Constrs Constr                              -- shorthand notation
  visit solve fixed                             -- a fixed visit (explained below)
    chn env      :: Map Ident IntSet            -- chained attribute
    syn changed :: Bool                         -- True if env changed

    default env       = head                    -- default rule for env
    default changed = or                        -- default rule for changed

sem Constr prod Subset                          -- approximation of loc.newVal
  loc.bVal       = lookupWithDefault ∅ loc.b lhs.env
  loc.aVal       = lookupWithDefault ∅ loc.a lhs.env
  loc.newVal   = loc.aVal ∪ loc.bVal
  lhs.env        = insert loc.b loc.bVal lhs.env
  lhs.changed = loc.newVal ≢ loc.bVal
```

Figure 2.10: AG for solving subset constraints.

explain some aspects of the example below.

We first give a grammar for a constraint language: a sentence in this language is a list of subset constraints ($a \subseteq b$) on some symbols (a, b) that represent integer sets. Given the list of constraints and an initial mapping *env* from symbol to integer set, we describe an algorithm that refines the mapping until all constraints are satisfied. The nonterminal *Constr* represents a subset constraint and the nonterminal *Constrs* a list of such constraints.

Secondly, we show how to refine the mapping for a single constraint, then show further below how to iterate over such a list of constraints. The rules in Figure 2.10 describe how the new *env* is computed from the initial *env* in a single iteration. The attribute *changed* is *True* if and only if the mapping was changed. The semantics for *Constrs* is fully determined by default rules.

Finally, we work below towards a specification of iteration for a list of constraints. The semantics of nonterminal *Constrs* is fully determined by the default rules: we thus specify iteration as part of the semantics of the Top-production, for which we introduce some additional annotation.

An invoke-rule may be annotated with strategies *z* as we saw earlier. We introduce two new strategies: *oneshot* and *iterate*. By default an invoke-ruke is (implicitly) annotated as oneshot, which means that the visit is at most invoked once. However, when the annotation is *iterate*, then the visit may be repeated multiple times:

```
r ::= invoke x of c̄ z          -- annotated invoke-rule
z ::= oneshot                  -- by default (implicit)
    | iterate e                -- repetitive invocation
```

```
again = Just    :: Inp N x → Maybe (Inp N x)   -- API function
stop  = Nothing :: Maybe (Inp N x)             -- API function
```

The expression *e* is a function that takes two parameters. The first parameter is the set of values for inherited attributes of the last iteration, and the second parameter the set of values for the synthesized attributes that resulting from that last iteration. The result of the function describes if the visit is repeated. If the result is produced using the constant *stop*, then the visit is not repeated. If, however, the result is produced using the function *again*, which takes as parameter the set of inherited attribute values that are used for the next iteration, then the visit is repeated with those values.

We apply this strategy to repeat the solve-visit on lists of constraints until a fixpoint is reached for the environment, which is the case when the attribute *changed* is *False* at the end of an iteration:

```
sem Top prod Top                              -- semantics of the root
  inh.root.env = lhs.initialEnv               -- env for the first iteration
  invoke solve of root iterate λ inp outp →   -- iterative invoke strategy
    if changed outp                           -- query attribute changed
    then again (inp {env = env outp})         -- repeat with updated env
    else stop                                 -- stop iterations
  lhs.result    = syn.root.env                -- takes result of last iteration
```

The values of the attributes are stored in a record for the inherited and synthesized attributes of a visit[11]. The labels are an encoding of the name of the attribute.

As a constraint solving strategy we may be interested in results of previous iterations. To keep a local state per node we introduce visit-local chained attributes, so that the notation for visit-declarations becomes:

$t ::= \textbf{visit}\ x\ \overline{\textbf{chn}\ y :: ty}\ \bar{r}\ t$ -- the type *ty* is optional

Note that the name of a visit may not clash with the name of a child, and *y* must be unique with respect to all visit-local attributes of a production.

The name *y* in the attribute declaration denotes four attributes that are local to the production:

attribute	meaning	scheduling	notation
inh.x.y	initial value of *inh.vis.y*	outside visit *x*	*inh* is a keyword
syn.x.y	last value of *syn.vis.y*	outside visit *x*	*syn* is a keyword
inh.vis.y	input to visit	inside visit *x*	*inh* and *vis* are keywords
syn.vis.y	result of visit	inside visit *x*	*syn* and *vis* are keywords

[11] In case of the generalization to phases of the previous section, an association of attributes to phases may be determined automatically when such an association is not manually given. However, when this happens, it is unclear which attributes are present in the record. Therefore, to be able to iterate a phase, we require the phase declaration to be annotated with the annotation *fixed* which disallows the automatic scheduling of attributes to the phase.

```
itf Constr                                               -- more visits
  visit initial      inh topVal  :: IntSet               -- value of top element
  visit solve fixed chn ...                              -- as defined above
  visit generate     syn outcome :: (IntSet, IntSet)     -- outcome of solving
sem Constr prod Subset
  lhs.output     = (loc.aVal, loc.newVal)                -- the computation for the result
  inh.solve.reps = 0                                     -- initial val of vis chained attr
  syn.vis.last   = inh.vis.last + 1                      -- increment
  ...
  loc.newVal     = loc.aVal ∪ loc.bVal ∪ loc.cVal
  loc.cVal       = if inh.vis.last ⩾ 5 then lhs.topVal else ∅
  visit solve chn reps :: Int                            -- visit local chained attribute
```

Figure 2.11: Example of weakening.

In Figure 2.11, we express that if the number of iterations exceeds a threshold of 5, then the result is weakened by enlarging it to the top-value in our set-lattice. This approach enforces convergence. The top-value is provided as attribute *lhs.topVal*. Note that we specified the rules outside the *solve* visit-block. The rule scheduling moves these rules to the appropriate block.

In contrast to conventional fixpoint evaluators for AGs, we precisely specify the iteration points, may perform fixpoint iteration over multiple attributes, and keep (purely functional) state between iterations. We may even construct children as part of a fixed visit, although to prevent constructing children over-and-over again, we may need to store the state of a child as part of the iteration state. We can accomplish this by detaching and attaching children.

First-class children. Constraints are used in inference algorithms to delegate a proof obligation to a different location in the tree. Constraints are typically used to delay a proof until all constraints in a given scope are collected.

In an AG, proof obligations can be encoded as a visit on a child that still needs to be invoked. Invoking the visit corresponds to constructing the proof. In the model as presented so far, we statically know the configuration of a child's state at each point during evaluation of the node. We present rules to detach and attach children that are in a given configuration, which permit us to treat children as first class values, and thus as constraints:

```
p = detach x of c    -- detaches c in the state that visit x is pending
attach x of c = e    -- attaches e as c in the state that visit x is pending
```

The attach-rule is a generalization of the child-rule that specifies that visit x and later are accessible on child c. The detach-rule specifies that visit x and later are not accessible on child c and provides a value that represents the child prior to the invocation of x. These two

```
itf Expr                                                  -- some example
  visit gather
    syn gathCnstrs :: IntMap Sem_Cnstrs_solve             -- use to gather of children
  visit distribute
    inh distCnstrs  :: IntMap Sem_Cnstrs_generate         -- used to distribute children
    syn transformed :: Expr
sem Expr prod Var
  child c : Constr      = sem_Subset loc.nm lhs.parentNm
  c.topVal              = lhs.topVal                      -- for first visit to c
  lhs.gathCnstrs        = singleton loc.nodeId (detach solve of c)
  attach generate of c  = lookup loc.nodeId lhs.distCnstrs
  lhs.transformed       = Expr_Const $ fst $ c.output      -- based on the last visit of c
```

Figure 2.12: Example of child detachment.

rules may be used in conjunction, however, to prevent conflicts only one attach or detach rule is allowed per child and visit combination. A detached child may be attached at a different location in the tree and visited as part of the evaluation for that location, or be visited through a wrapper function in the host language as part of an external solving algorithm.

Figure 2.12 shows an example of how a child can be detached. We collect the detached children in an attribute *gathCnstrs* as constraints. These constraints are solved elsewhere by invoking the *solve* visit on them, then transferred back as attribute *distCnstrs* and attached again. The type *Sem_Cnstrs_solve* is the type of a detached child prior to the invocation of *solve*. At another location in the tree, we may attach the constraints in the list and apply the iteration technique of above to solve the constraints.

This approach has the advantage that we can easily transport context information from the node that defines the constraint to the location where we solve the constraint, and vice versa. Middelkoop [2011b] gives additional examples of this technique. Moreover, the dependency analysis provides define-before-use guarantees. Chapter 9 describes how dependent types can be used to prove that a detached subtree is attached at precisely one other location in the tree.

Remarks. By making visits explicit we gained the ability to describe evaluation strategies by annotating the callee (visit declarations) or the caller (invoke-rules). Clauses offer a means to specify alternatives. Children in a given state are first class and can be passed around to describe complex traversals. The extensions preserve the attractive properties of AGs, such as automatic rule scheduling and purely functional descriptions. Also, the implementation in the host language is purely functional.

In Chapter 5 we present a large example, and give a translation of the notation to Haskell code. We show in Middelkoop [2011b] that we can also describe complex traversals over trees and even graph structures.

```
data Index a where       -- data type written using GADT notation
  TInt :: Index Int       -- parameterless constructor with a ≡ Int
  TBool :: Index Bool     -- parameterless constructor with a ≡ Bool
append :: Index a → a → a → a
append TInt  = (+)                                   -- coercion of Int to a
append TBool = (∧)                                   -- coercion of Bool to a
ex1 = (append TInt 1 2, append TBool True False)     -- OK: (3, False)
ex2 = append TInt 1 2 + append TBool True False      -- type error (Int != Bool)
ex3 = append TBool 1 2                               -- type error (Int != Bool)
```

Figure 2.13: Example of a GADT as type index.

2.4 Case Study with GADTs

Constructors of an algebraic data type specify how a value of the data type is structured. A data type may be parameterized. Generalized Algebraic Data Types (GADTs) [Cheney and Hinze, 2003] associate per data constructor a set of type equivalences between the parameters of the data type. When building a value using a GADT constructor, and thus specifying how the parameters are instantiated, the type equivalences must be satisfied. In the scope of a successful pattern match against a GADT data constructor, the type equivalences may be assumed to hold and can be used to *refine* or safely *coerce* types.

In the extended edition of this thesis, we present a type system for GADTs as a case study for several reasons [Middelkoop, 2011a]. Firstly, a type system for GADTs poses additional challenges to a description of a type inference algorithm compared to a conventional DHM-style inference algorithm (Section 1.2.6, Section 1.3.11) which give insight in what features our meta language for type system needs to support. Secondly, we investigate the description of GADTs as a minimalistic type system *extension*. Moreover, we make extensive use of GADTs in this thesis, thus this chapter can then also be used as an explanation of GADTs.

In this section, we take a simplified subset of the actual type system: equality proofs. We first give a specification, then look at properties of an inference algorithm, and consider a description of the algorithm with attribute grammars.

Specification. GADTs are typically used as type index. In the example of Figure 2.13, the type *Index a* is a first-class description of the type *a*. By pattern matching on the description we reconstruct what the concrete type was to which *a* was instantiated. The expression *TBool* must be of type *Index Bool*, thus the *a* of *append TBool* must be *Bool*. Consequently, ex_2 and ex_3 are ill-typed.

In the above example, there exists a type equality assumption $a \sim Int$ in the context of having matched against the *TInt* constructor. The assumption is used to prove that $Int \to Int$ can be coerced into $a \to a$. The actual facilities that we need to reason with GADTs is the introduction of type equality assumptions in a scope, and equality reasoning on types. These

$$\boxed{\Gamma \vdash \tau \equiv \rho}$$

$$\Gamma \vdash \tau \equiv \tau \ \textsc{refl} \qquad \frac{\Gamma \vdash \tau \equiv \rho}{\Gamma \vdash \rho \equiv \tau}\ \textsc{sym} \qquad \frac{\Gamma \vdash \tau \equiv \rho \quad \Gamma \vdash \rho \equiv \sigma}{\Gamma \vdash \tau \equiv \sigma}\ \textsc{trans} \qquad \frac{(\tau \sim \rho) \in \Gamma}{\Gamma \vdash \tau \equiv \rho}\ \textsc{assum}$$

$$\frac{\Gamma \vdash \tau \equiv \rho \qquad \Gamma \vdash \sigma \equiv \omega}{\Gamma \vdash \tau \rightarrow \rho \equiv \sigma \rightarrow \omega}\ \textsc{congr} \qquad \frac{\tau \rightarrow \rho \equiv \sigma \rightarrow \omega}{\Gamma \vdash \tau \equiv \sigma}\ \textsc{sub.l} \qquad \frac{\tau \rightarrow \rho \equiv \sigma \rightarrow \omega}{\Gamma \vdash \rho \equiv \omega}\ \textsc{sub.r}$$

Figure 2.14:

facilities are orthogonal to the actual treatment of algebraic data types. To be able to describe GADTs as a separate aspect of a type system, it is thus desirable to separate these facilities.

In the above example, we used simple types constructed by type arrows and type constants. In the specification we use the following grammar for types and environments containing equality assumptions:

$\tau ::= a \mid \mathit{Int} \mid \mathit{Bool} \mid \tau_1 \rightarrow \tau_2$	-- types, also: ρ, ω, and σ
$\Gamma ::= \emptyset \mid \Gamma, (\tau_1 \sim \tau_2)$	-- environment containing type equality assumptions

The type equality relation is used to reason with the equality between types.

Given an environment Γ that consists of the type equality assumptions, the inference rules in Figure 2.14 describe the type equality relation. The first three rules are properties that any equality relation is supposed to exhibit. In addition, the rule ASSUM expresses a proof by assumption, and the remaining three are related to congruence and subsumption properties derived from the structure of types. See Middelkoop [2011a] for some exemplary proofs for judgments of this relation.

Forward and backward chaining. The above rules are not straightforwardly mapped to an inference algorithm. The rule SYM can always be applied, thus some condition is needed to determine when not to apply this rule. In Section 1.2.6 we describes properties of the type rules of the DHM type system that permit an attractive implementation in the form of algorithm W. Similarly, we apply apply domain knowledge here to impose conditions on above rules so that problematic derivation trees are avoided or do not have to be considered by the algorithm. For example, we require that a derivation tree for judgment $a \equiv b$ may not contain (indirectly) a child for the same judgment, as the proof would then be circular. This constraint ensures that the number of applications of the sym-rule is bounded.

To explain why the other rules are not straightforwardly mapped to an inference algorithm, we mention first that there are two ways of reasoning with inference rules [Russell et al., 1996]. With *forward chaining* inference starts with assumptions and derives conclusions. With *backward chaining* inference starts at conclusions and tries to prove premisses until they can be discharged by assumptions. Inference algorithms as discussed so far use a limited form of backward chaining.

Backward chaining is suitable when a conclusion can be decomposed into smaller premisses, which is indeed the case for rules REFL, ASSUM and CONGR, and also for rule SYM with the aforementioned restriction. This is not the case for the rules TRANS, SUB.L, and SUB.R. These contain one or more meta variables in their premises that are not fixed by their conclusion judgments. As a consequence, arbitrary (infinite) branches can be introduced in the derivation tree by applying these rules.

Forward chaining is suitable when premisses can be decomposed into smaller conclusions. This is the case for all rules except REFL and CONGR. To deal with all rules, we use a combination of backward and forward chaining by distinguishing proof obligations and proven facts. Rule CONGR may only be applied on a proof obligation whereas Rules SUB.L and SUB.R may only be applied on proven facts. The TRANS may only be applied if one of the premisses is a proven fact. As part of the case study in Middelkoop [2011a], we implemented a solver for equality constraints in UHC using an implementation of constraint handling rules [Frühwirth, 1998] that provides forward chaining and can emulate backward chaining [Dijkstra et al., 2007b].

Lookahead. Forward chaining can be implemented with backward chaining by defining a reduction relation on environments which keep track of derived facts. We thus concern ourselves in the remainder of this section with the implementation of backward chaining using attribute grammars.

To implement backward chaining, we use clauses to represent the various alternatives. However, the rule TRANS poses an additional challenge: a choice made for the left premise has consequences for the right premise. To express that a clause may only be selected if the remaining evaluation in a given context (the remainder-context) does not fail, we introduce two more strategy annotations: the lookahead-strategy and the onlylocal-strategy which serve as annotations for visit-blocks and invoke-rules:

$$
\begin{array}{lll}
t ::= & \textbf{visit}\ x\ z\ \overline{k} & \text{-- visit-block as presented before} \\
r ::= & \textbf{invoke}\ x\ \textbf{of}\ \overline{c}\ z & \text{-- invoke-rule as presented before} \\
z ::= & \textbf{onlylocal} & \text{-- does not take the remainder-context into account} \\
\mid & \textbf{lookahead} & \text{-- takes the remainder-context into account}
\end{array}
$$

The onlylocal-strategy is the default.

The remainder-context is a runtime property that can be influenced by invoke-rules. When an invoke rule is evaluated and it is annotated with the lookahead-annotation, the remaining evaluation in the parent's remainder context contributes to the remainder-context of the child. Otherwise, the invoke-rule behaves as a cut-operator which fixes the choices of clauses made by the child. The partial-strategy and total-strategy as mentioned before are orthogonal to the onlylocal-strategy and lookahead-strategy.

The example in Figure 2.15 serves as illustration and we explain it below. For brevity, we left out details of the description that are related to the prevention of infinite derivations, the administration of substitutions, and the reuse of prior derivation trees. Moreover, we

```
grammar Eq prod Check
itf Eq visit check partial
  inh env        :: Set (Ty, Ty)
  inh tp_1, tp_2 :: Ty
sem Eq prod Check visit check
  lookahead        -- visit annotation
  default env      -- rule in scope visit
  invoke check lookahead parent
  clause refl
    child u : Unify = sem_Unif
    u.tp_1  = lhs.tp_1
    u.tp_2  = lhs.tp_2
  clause sym
    child flipped : Eq = sem_Check
    flipped.tp_1 = lhs.tp_2
    flipped.tp_2 = lhs.tp_1
  clause trans
    child fr    : Fresh = sem_Fresh
    child left  : Eq    = sem_Check
    child right : Eq    = sem_Check
    left.tp_1  = lhs.tp_1
    left.tp_2  = fr.tp
    right.tp_1 = fr.tp
    right.tp_2 = lhs.tp_2
  clause assum
    match (u_1.tp_2, u_2.tp_2) ← do
      ahead $ λk → some $
        map k $ elems lhs.env
    child u_1, u_2 : Unify = sem_Unif
    u_1.tp_1 = lhs.tp_1
    u_2.tp_1 = lhs.tp_2
```

Figure 2.15: Example of an AG that represents an equality solver.

omitted attributes and rules to construct coercion terms from such a derivation tree. Such topics are discussed in Section 2.3. Only one production is declared for the Eq-nonterminal. The clauses in combination with higher-order children determine the structure of the equality proof.

The example features a monadic match rule. The right-hand side of this rule is a monadic expression that determines the value to match against. In clause *assum*, we take a type equality assumption from the environment. There may be multiple of such assumptions in the environment. We derive from these assumptions a monadic expression that explores the possibilities one after the other and selects the first one that succeeds. Via *ahead* (explained below), we get a continuation *k* that expects a value for the pair and performs the remaining computations for the current context. The function *some* is defined below. It selects the first computation that succeeds.

We saw above how to express backward chaining with clauses in combination with lookahead. In Section 2.1 we mentioned that the AG can be expressed as a monad. First we show an implementation of lookahead by using a backtracking monad, then show how clauses can be mapped to this monad.

Backtracking monad. We wrap the actual underlying monad *m* into a monad transformer *BackT* that consists of a composition of the continuation transformer on top of the error transformer. The continuation monad transformer provides a continuation, and via the error monad transformer a failing computation can be observed [Jones, 1995]. The result type of the continuation is the parameter *r*:

```
data Back = Back                                -- backtrack message
type BackT r m a = ContT r (ErrorT Back m) a    -- transformer
```

Backtrack points are specified using the operator (⦶) which represents local choice. It selects its right argument if and only if the evaluation of the left argument fails. Alternatively, the operator ⊕ represents a global choice, which takes the continuation of the parent of the choice into account:

```
(⦶) :: Monad m ⇒ BackT a m a → BackT a m a → BackT r m a
p ⦶ q = ContT (λc → catchError (cut p) (const (cut q)) ≫= c)
(⊕) :: Monad m ⇒ BackT r m a → BackT r m a → BackT r m a
p ⊕ q = ConT (λc → catchError (runContT p c) (const (runContT q c)))
cut p     = runConT p return
msum      = foldr (⦶) (fail "backtrack")
resolve p = ContT (λc → cut p ≫= c)
```

The function *resolve* limits the continuation. The function *ahead* exposes the continuation to the higher-order function *f*:

```
ahead :: Monad m ⇒ ((a → ContT r m r) → ContT r m r) → ContT r m a
ahead f = ContT (λc → runContT (f (λa → ContT (λk → c a ≫= k))) return)
p ⊕ q   = ahead (λk → p ≫= k ⦶ q ≫= k)    -- alternative implementation
```

The function *ahead* provides the ability to explore different values for the continuation, and make choices based on the outcome of the continuation. We show in Chapter 7 how to extend this mechanism to make choices based on intermediate results that are computed in the continuation.

In a continuation monad, a computation *BackT r m a* represents a computation for a value of type *r* with a pending computation that takes *a* to *r*. The function *f* in *ahead* takes the pending computation as parameter, and replaces the computation for *r* with a computation that immediately goes to *r*. *Ahead f* can thus be understood as replacing the pending computation with (the computation produced by) *f*.

Mapping of clauses to monads. The evaluation algorithm for a clause is a monadic expression that computes values for the synthesized attributes of the visit. We thus define the body of a visit function as a sequence of these monadic expressions that are either combined with the global choice operator when the visit is annotated with the lookahead-annotation, or with the local choice operator when the visit has the default onlylocal-annotation. If an invoke-rule is not annotated with a lookahead-annotation, it applies *resolve* to the monadic expression of the child after applying the values for the inherited attributes.

Remarks. As mentioned in the previous section, clauses represent a search tree, which encodes alternative ways to compute the decorations of the tree. The exploration of these alternatives using the *BackT* monad is depth-first. Chapter 7 describes how to explore clauses in a breadth-first way, which may give a more balanced exploration.

Overhead is the work that is performed for the exploration of an alternative that is not selected. In practice, we preferably solve problems using a single pass traversal, or a fixpoint iteration. A search for a solution, however, cannot always be avoided, as is demonstrated by the GADT use-case. Moreover, the naive exploration of alternatives may be convenient for prototyping purposes.

2.5 Attribute Grammars with Stepwise Evaluation

Some type inference algorithms require an exploration of a forest of potential derivation trees. We can encode such a forest as a search tree that contains additional nodes which represent choices between derivations. In Chapter 7 we present a library to describe such explorations of the search tree.

Stepwise evaluation. In the evaluation algorithms of Chapter 5, clauses are explored one after the other. This approach corresponds to a depth-first exploration of alternatives. In Chapter 7 we show how to evaluate clauses simultaneously, which corresponds to a breadth-first exploration of alternatives. A breadth-first exploration provides a balanced exploration for alternatives, which may be more efficient.

With statically ordered AG evaluation (Section 1.3.4), the evaluation of an AG is a sequence of rule evaluations. In this section, we group a number of these rule evaluations together and call that a step. We represent the evaluation of a tree as a computation which can be asked to execute one step, and afterwards pauses and returns control back to the caller. To decorate the tree, we provide a computation (the root-computation) at the root which takes the computation of the tree and repeatedly asks it to perform a step until the decorations are computed.

Simultaneous exploration. To explore alternatives, we mentioned in the previous section that we combine the computations of alternatives, for example using the $\oplus$-operator. We now consider different ways to combine the computations of alternatives. We provide a computation (the choice-computation) that asks the alternatives to perform steps in an interleaved fashion. When an alternative succeeds, we replace the choice-computation with the alternative. When an alternative fails, we replace the choice-computation with the other alternative. When each alternative performed one step, the choice-computation exposes one step to its parent choice-computation or the root. With this approach we obtain a breadth-first traversal of alternatives.

In this section, we first describe how to write such an algorithm as a monad that represents a *coroutine*, and how to specify what constitutes to a step in this monad. Then, we show how this monad is used in RulerCore descriptions, and show an implementation of the monad.

Coroutines. A coroutine is a function that during its execution performs zero or more *yield* operations which denote re-entry points. A *yield* operation pauses the execution of the function and returns control to the caller. The caller may resume the execution of the callee from the point where it was paused. The callee may expose intermediate results to the caller

```
yield  :: Stepwise m ()
step   :: Monad m ⇒ Stepwise m a → m (Report m a)
lift   :: Monad m ⇒ m a → Stepwise m a
ahead  :: (∀r.(a → Stepwise m r) → Stepwise m r) → Stepwise m a

data Report m a                 -- represents a progress report
   = Done a                     -- finished and produced a value a
   | Failed  String             -- failed with a given error message
   | Paused (Stepwise m a)      -- paused with the residual computation
```

Figure 2.16: API of the Stepwise monad.

and the caller may provide additional parameters when resuming the function. We assume initially as simplification that no results are exchanged between caller and callee.

Visit functions are examples of coroutines (Section 1.3.5) that are invoked a statically fixed number of times. The evaluation of a child pauses at the end of the visit, and proceeds with the evaluation of the next visit when the parent invokes the subsequent invoke-rule. In this section, however, we consider coroutines that in addition to the statically fixed yields between visits, may yield a statically unbounded number of times during the execution of a visit. The evaluation up-to the next yield is what we call a step.

We design a coroutine monad *Stepwise*, which represents a stepwise computation specialized for the exploration of alternatives. It supports a number of operations in addition to those of the *BackT* monad. Figure 2.16 shows the API. The operation *yield* pauses the execution and resumes the caller. The operation *step* runs the coroutine until the coroutine either fails or succeeds, or reaches the next yield instruction. The outcome of the evaluation is presented as a progress report in the encapsulated monad *m*. A yield-operation thus specifies what constitutes as a single step. With *lift*, we wrap the effects of *m* into a stepwise computation so that these effects can be merged with the effects embedded in other stepwise computations. For example, we typically use *lift* to describe how the effects of a step-operation on a child are merged with some parent computation.

With the choice combinators, we define a computation that represents a traversal over a search tree. Each subtree encodes an alternative. With the above API, Figure 2.17 shows a breadth-first version of the choice combinators. The traversal is breadth-first because when *act* reports a step to the caller, each of the non-failed children performed one step. Iteration is encoded by replacing the choice-computation with a computation that calls the choice function again. Thus, when we commit to a certain alternative, we replace the choice-computation with the selected alternative, and thereby eliminate the choice.

Figure 2.18 A depth-first version of the choice combinators is obtained by applying the function *fullred* to the left alternative. This function returns the computation with all steps stripped, thus forcing it to evaluate fully. The control we have over stepwise computations allows us to express a whole range of strategies, such as taking two steps left for each step right.

```
p ⊕ q = ahead (λk → p >>= k ⦶ q >>= k)
p ⦶ q = do a ← lift (step p)                 -- perform a step for p
           b ← lift (step q)                 -- perform a step for q
           act a b                           -- inspects the outcomes

act :: Report m a → Report m a → Stepwise m a
act (Done a)    _           = return a        -- commit to finished p
act _           (Done a)    = return a        -- commit to finished q
act (Failed s)  (Failed _)  = fail s          -- both fail
act (Failed _)  (Paused r)  = r               -- p fails, commit to q
act (Paused r)  (Failed _)  = r               -- q fails, commit to p
act (Paused p') (Paused q') = yield ≫ (p' ⦶ q')   -- pause, later continue with choice
```

Figure 2.17: Breadth-first choice combinators.

```
fullred :: Stepwise m a → m (Stepwise m a)
fullred p = do rep ← step p                  -- perform a step
               case rep of                   -- inspect report
                 Paused r → fullred r        -- repeat after yield
                 _        → return $ comp rep   -- either Failed or Done

comp :: Report m a → Stepwise m a    -- report to residual computation
comp (Paused m) = m
comp (Failed s) = fail s
comp (Done v)   = return v
```

Figure 2.18: Depth-first choice combinators

The underlying monad *m* can be used to exchange information between the computation of an alternative and the choice between alternatives. For example, when *m* is a writer monad, an alternative can provide an estimate of the amount of work that has been performed. When *m* supports IO, the system time can be used to balance the two computations. Stepwise computations thus offer a means to describe powerful and complex exploration strategies.

Children as stepwise computations. The evaluation of an AG we represent as a monadic computation, and thus fits the Stepwise-monad straightforwardly. In the remainder of this section, we show how to specify yield operations and how to express alternatives.

We do not need to introduce additional syntax to express yield operations because monadic child rules have monadic right-hand sides and can therefore be used to express the yield operations. For example, we introduce a dummy nonterminal *Yield* in Figure 2.19 and use it to specify a yield operation using a child rule in some exemplary production *Var*. A monadic

```
itf Yield                        -- interface without visits nor attributes
grammar Yield prod Yield         -- single production
sem Yield prod Yield             -- empty semantics
sem Expr prod Var visit check
   child y : Yield ← do          -- monadic child rule
      yield                      -- monadic operation
      return sem_Yield           -- semantics of child (trivial)
```

Figure 2.19: Yielding of steps expressed as nonterminal.

```
sem Tree prod Alt                         -- semantics for a choice node
   child left  : Tree = ...               -- define child left
   child right : Tree = ...               -- define child right
   left.i₁            = ...               -- definition of some attributes
   right.i₁           = ...
   loc.p = detach upon v of left          -- evaluates left up to visit v
   loc.q = detach upon v of right         -- evaluates right up to visit v
   attach upon v of res : N ← do          -- attaches as child res
      loc.p ⊕ loc.q                       -- choice between children
   lhs.s  = res.s₁                        -- use results of the chosen child
```

Figure 2.20: A sketch of a parallel exploration.

child rule is guaranteed to be evaluated in the visit it is constrained to. Thus, the right-hand side of the rule is evaluated during visit *check*.

Encoding of alternatives. To express alternatives we can explicitly encode a search tree using higher-order children or by using clauses. We first consider the encoding of a search tree. A node in a search tree may express a choice between its children. For this purpose we refine the notation introduced in Section 2.3 to attach and detach children with an additional keyword (explained below).

We detach alternatives and attach a computation that determines the chosen alternative. The abstract example in Figure 2.20 provides a sketch. Recall that the semantics of a visit of a child is a function that takes values for inherited attributes and returns a monadic computation for the synthesized attributes of that visit. We are thus interested in the state of the children after they received the values for the inherited attributes. We use the upon-keyword for this purpose. The detach-rule thus provides the monadic computation for the specified visit of a child, and the attach-rule runs the computation to obtain the synthesized values for that visit.

```
data Stepwise m a where
  Return  :: a → Stepwise m a
  Bind    :: Stepwise m a → Parents m a b → Stepwise m b
  Fail    :: String → Stepwise m a
  Yield   :: Stepwise m ()
  Lift    :: m a → Stepwise m a
  Ahead   :: (∀r.(a → Stepwise m r) → Stepwise m r) → Stepwise m a
data Parents :: m a b where
  Root    :: Parents m a a
  Pending :: (a → Stepwise m b) → Parents m b c → Parents m a c
instance Monad (Stepwise m) where
  return  = Return
  m >>= f = Bind m (Pending f Root)
```

Figure 2.21: The structure of *Stepwise*.

Strategy for clauses. Instead of explicitly encoding a search tree as above, we can also use clauses. To specify a choice between clauses, we present additional notation. A visit-block may be annotated with a select-strategy:

```
t ::= visit x z c̄     -- existing syntax for visit-blocks
z ::= onlylocal       -- combine clauses with the local choice operator
   |  lookahead       -- combine clauses with the global choice operator
   |  select e        -- custom, e is e.g. a function λc₁ ... cₖ → ...
```

The select-strategy specifies a function e that takes a computation for each clause of the visit-block as parameter and provides a computation for the results of the visit. The computation for the results of a visit is $e\ c_1 \ldots c_k$ where $c_1, \ldots, c_k$ are the computations corresponding to each clause[12].

Implementation. In the remainder of the section we describe how intermediate results can be exposed to the selection function. However, we first show how the Stepwise-monad is implemented. The algorithm[13] that we show here is slightly simplified with respect to Chapter 7 and its full understanding is not required for the remainder of this chapter.

We represent the Stepwise-monad in Figure 2.21 as a computation that can be inspected (Section 1.3.9). The function *step* interprets the computation in order to evaluate it one step. The right-hand side of a bind contains a stack *Parents* of all the continuations to the right of a monadic expression. The expression $(m \gg\!\!= f) \gg\!\!= g$ is represented as:

[12] Several notational variants are possible. It may sometimes be more convenient to obtain the computations of the clauses as a list or as a record with a field for each clause.

[13] Complete Haskell module of the simplified implementation: `https://svn.science.uu.nl/repos/project.ruler.papers/archive/RefStepwise.hs`

```
step :: Monad m ⇒ Stepwise m a → m (Report m a)
step m                              = reduce m Root

reduce :: Monad m ⇒ Stepwise m a → Parents m a b → m (Report m b)
reduce Yield        r               = return $ Paused (r `apply` ())
reduce (Fail s)     _               = return $ Failed s
reduce (Lift m)     r               = m >>= step.apply r
reduce (Ahead f)    r               = step $ f (apply r)
reduce (Return v)   Root            = return $ Done v
reduce (Return v)   (Pending f r)   = reduce (f v) r
reduce (Bind m r)   r'              = reduce m (push r r')

apply :: Parents m a b → a → Stepwise m b
apply r v                           = Bind (Return v) r

push :: Parents m a b → Parents m b c → Parents m a c
push r            Root              = r
push Root         r                 = r
push (Pending f r') r               = Pending f (push r' r)
```

Figure 2.22: The implementation of *reduce*.

Bind m (Pending f (Pending g Root))

Since monadic binds are right-associative, the stack only grows when an expression occurs as left-and side of a monadic bind that expands to one or more binds. This is the case when calling the visit function of a child, hence the stack contains the continuations of all parents till the location where *step* is performed. The child that is undergoing evaluation is on top of the stack.

The function *reduce* in Figure 2.22 takes a computation *m* and a pending stack *p*. It evaluates *m* one step. If *m* yields or fails it returns a progress report. Otherwise, it continues evaluating *m* until it obtains a result that can be fed into the parents-stack. If the parents-stack is empty, evaluation is finished. Otherwise, the parent-stack contains the continuation to proceed with. The function *step* delegates to *reduce* with an empty stack. The function *apply* turns a pending parent into a monadic computation by passing it the result it was waiting for. The function *push* concatenates two parent stacks.

Coordination. We presented above how to specify selection strategies. To encode powerful coordination strategies, we improve the above approach so that computations can yield results and take arguments when resumed.

Definition (Tag)**.** A *tag* of type *Op i o* specifies the interface between the callee (the computation) and the caller (execution of *step*), where *i* is a type index that fixes the data-type that is used for tags (*Op i o* is a type family), and *o* is the type of the results exchanged between

```
action :: Op i o → Inp o → Stepwise m (Out o)
data Report i m a              -- refinement of the Report data type
   | ∀o.  Paused   (Op i o)  (Inp o)
                   ((Stepwise m (Out o) → Stepwise m a) → Stepwise m a)
data family Op i  :: ∗ → ∗     -- a tag of an operation (of some set indexed by i)
type family Inp o :: ∗         -- inputs to the operation Op i o
type family Out o :: ∗         -- resulting outputs of operation Op i o
```

Figure 2.23:

callee and caller. The computation yields results of the type *Inp o* and takes arguments of type *Out o* as parameter for the next visit.

We refine the operation *yield* so that it takes a tag of type *Op i o* and intermediate results of type *Inp o* for the caller and provides arguments of type *Out o* as given by the caller when the callee is resumed. Figure 2.23 shows the encoding in Haskell with type families.

For example, *action* can be used by a computation to report the number of open goals, and receive a priority rating from the caller. In that case, we specify the following instances for the above type families:

```
data Meta                        -- type index for the set of tags
data OInfo                       -- type index for a particular tag
data instance Op Meta o where    -- declares the tags
   OpInfo :: Op Meta OInfo       -- one tag
type instance Inp OInfo = Int    -- specification of inputs of operation o
type instance Out OInfo = Int    -- specification of results of operation o
```

The report-handling code in selection strategies may match on Yield-reports to obtain the *Op i o* and *Inp o* values, plus the continuation which may be used to resume the computation when a computation is provided that produces the *Out o* values.

This approach is powerful: arbitrary *traps* to operations can be expressed this way. For example, it is possible to create tags for that represent operations such as unification, generation of unique numbers, lookups in memo tables, and output to the console. The caller can determine what semantics to give to these operations. The construction offers an *inversion of control*: the callee *declaratively* specifies operations, and the caller determines the semantics.

Memoization. As mentioned in the GADT example, it may be desirable to cache results of subtrees and reuse these results at other locations in the tree. However, if the continuation (accessed using *ahead*) is used to distinguish the result of the computation, the results should not be cached, unless the continuation is the same for each context the shared computation appears in.

Moreover, in case of a stepwise computation, these results may not be available yet, and it may be desirable to share computations instead. A computation can be shared by storing it in an updatable state, and updating this state after each call to *step*. The computation may then each time receive the result of an action from a different context. Also, each contexts may only receive a partition of the actions yielded by the computation.

Remarks. Chapter 7 focusses on *generators* which are coroutines that only yield information but do not take parameters. We show how to evaluate such coroutines strictly or lazily. In the latter case, results can already be produced when it depends on a choice for which only one alternative is left.

2.6 Attribute Grammars with Dependent Types

In Chapter 9 we investigate AGs where attributes may have a dependent type, which can be used to state and prove properties of the AG. For this purpose, we describe an embedding of AGs in Agda[14] [Bove and Dybjer, 2009]. *Dependent types* provide a means to use types to encode properties with the expressiveness of (higher-order) intuitionistic propositional logic, and terms to encode proofs. In this setting, a parameterized type constructor specifies a relation between its type parameters and data constructors form the inference rules of the relation.

In a dependently typed AG, the type of an attribute may refer to values of attributes. The type of an attribute is an invariant and the value of an attribute a proof for that invariant. Moreover, because of the Curry-Howard correspondence, dependently typed AGs are a domain-specific language to write structurally inductive proofs in a *composable*, *aspect-oriented* fashion; each attribute represents a separate aspect of the proof.

Some knowledge of dependent types and Agda is a prerequisite for this section. We first give an example and some notation. We follow up with an extension that permits the visits of a nonterminal to be organized as a tree instead of a totally ordered sequence.

Dependent attribute type. The following interface declaration for some nonterminal *Pat* demonstrates attributes with a dependent type. The gathered environment *syn.gathEnv* is a subset of the final environment *inh.finEnv*, which is expressed as the type *syn.gathEnv* ⊆ *inh.finEnv*. We assume the existence of a type constructor ⊆ and several utility functions. The attribute *inh.gathInFin* has the above type and therefore represents a proof of this property:

```
itf Pat
  visit analyze    syn gathEnv   :: Env
  visit translate  inh finEnv    :: Env
                   inh gathInFin :: syn.gathEnv ⊆ inh.finEnv
grammar Pat   prod Var   term nm :: Ident
```

[14] In this section, we deviate slightly from the actual syntax to have a closer correspondence with Haskell.

The *gathInFin* attribute provides the guarantee that elements that are in *syn.gathEnv* are also in *inh.finEnv*. Rules of productions of *Pat* may exploit this guarantee.

The *lookup* of an identifier in the final environment may return *Left notIn* where *notIn* is a proof that the identifier is not in the environment, or *Right v* where *v* is the value of the identifier in the environment.

In the following example, production *Var*, which has a terminal *loc.nm*, we define with a proof[15] $loc.prv_2$ that the identifier is in the environment, and we use $loc.prv_2$ to prove that the lookup cannot return a Left-value:

```
sem Pat prod Var
  loc.prv_1 = here loc.nm syn.lhs.gathEnv          -- proof of nm in gathEnv
  loc.prv_2 = inSubset lhs.gathInFin loc.prv_1     -- proof of nm in finEnv
  loc.val   case loc.nm ‘lookup‘ lhs.finEnv of     -- defined by case distinction
              | Left notIn   falsum notIn loc.prv_2 -- impossible case
              | Right v        → v                  -- case that loc.nm is in finEnv
```

The case that the element is not in *lhs.finEnv* is in contradiction with $loc.prv_2$. Their application has an uninhabitable type, which we use in combination with the falsum-case to terminate the branch without giving a definition.

This example shows three ways to define an attribute: with a plain RHS, with case distinction, and with **falsum** *e* where host-language expression *e* has an uninhabitable type:

```
r ::= p = e          -- with plain RHS
    | p m            -- with complex RHS
m ::= falsum e       -- unreachable case (e has an uninhabitable type)
    | case e of b̄    -- with case distinction, and cases b̄
b ::= ρ → e          -- nested plain RHS in a case distinction
    | ρ t            -- nested complex RHS in a case distinction
ρ                    -- pattern in the host language (no attributes)
```

The additional syntax provides us with a means to give provable *total* definitions of attributes.

Cycle analysis and consistency. Functions are required to be total in dependently typed programs for reasons of logical consistency and termination of type checking, which in case of AGs correspond to total definitions of attributes and the requirement that dependencies between attributes are acyclic.

Partitions. In Chapter 4, the ordering algorithm implicitly distinguishes different contexts in which a nonterminal is used. However, to ensure that attribute definitions are total, it may be convenient to make such contexts explicit.

In the following example, we specify code generation depending on the absence of errors. We define two contexts for the *generate* visit. The context *errorfree* provides an attribute

[15] The functions *here* and *inSubset* are conventional dependently typed functions that construct the appropriate proofs. Their implementation are beyond the scope of this section.

```
sem Pat prod App
  lhs.errors = f.errors ++ a.errors                 -- collect errors
  context errorfree                                 -- rules exclusive for errorfree
    invoke generate of left   context errorfree     -- specifies context to invoke
    invoke generate of right  context errorfree     -- specifies context to invoke
    left.noErrors  = leftNil   left.errors right.errors lhs.noErrors
    right.noErrors = rightNil  left.errors right.errors lhs.noErrors
    lhs.code = left.code `apply` right.code
    context haserrors                               -- rules exclusive for haserrors
      invoke generate of left   context haserrors
      invoke generate of right  context haserrors
      lhs.pretty = left.pretty ⊕ right.pretty       -- collect pretty print
```

Figure 2.24: Example of rules specified for a specific context.

code, but it may only be invoked when errors are absent. The context *haserrors* alternatively provides an attribute *pretty*, which contains an annotated pretty print of the program:

```
itf Pat
  visit report   syn errors :: Errs inh.finEnv
  visit generate          -- a visit may consist of one or more partitions
    context errorfree     -- a partition has a name
      inh noErrors :: syn.errors ≡ [ ]
      syn code     :: Target inh.finEnv
    context haserrors     -- a partition may also contain subsequent visits
      syn pretty   :: Doc
```

The caller invokes a visit on the callee, and is responsible for selecting what context it wants to use. The callee is required to produce results for that choice. The callee can encode restrictions on the available choices for the parent as inherited attributes. The caller must provide values for the inherited attributes of the partition it chooses.

We specify a context as annotation of the invoke-rule. Moreover, we may specify rules for particular contexts as is demonstrated in Figure 2.24. A special falsum-rule may be used to denote that a visit, clause or context is unreachable.

Remarks. Type attributes correspond to quantification. An inherited type attribute corresponds to universal quantification, since the caller can choose its instantiation. A synthesized type attribute corresponds to existential quantification. The callee can choose its type, but the caller cannot make an assumption about it. This mechanism allows us to deal with polymorphism in interfaces.

Indeed, the above ideas allow quantification in an AG for Haskell to be expressed. In a dependently typed AG, attributes can represent both values and types. In Haskell, there is a clear distinction between values and types. In an AG for Haskell, we can make an explicit distinction between attributes that represent types (and specify a *kind* as type) and attributes that represent values. The type of a type attribute may not refer to other attributes. The type of a value attribute, however, may refer to a type attribute.

2.7 Attribute Grammars on DAGs

In the extended edition, we included a relatively short chapter [Middelkoop, 2011a] that provides examples of other ways to apply the techniques as presented. There are two common data structures in compilers: trees and directed graphs. Ordered attribute grammars are suitable to define a semantics on trees but not suitable to define the semantics of graphs. The reasons is that nodes in a graph may occur in different contexts at execution time, which makes a static dependency analysis difficult. In that chapter, we also show how our approach relates to (cyclic) reference attribute grammars.

2.8 Conclusion

We gave a detailed summary of the following chapters, and described how the chapters are connected together. Also, this chapter showed the features of RulerCore in relation to conventional attribute grammars.

A prototype implementation of RulerCore is available as the compiler `ruler-core`. Its implementation is based on higher-order attribute grammars and Haskell, and can be obtained from:

```
https://svn.science.uu.nl/repos/project.ruler.papers
       /archive/ruler-core-1.0.tar.gz
```

The `examples` subdirectory contains some minimalistic examples. A large example based on the HML type system [Leijen, 2009] can be obtained from:

```
https://svn.science.uu.nl/repos/project.UHC.pub
       /branches/tnfchris-hml/
```

3 AGs with Side Effects

This chapter introduces the concept of visits, which play an important role in subsequent chapters of this thesis. We present this concept by means of a correspondence with the visitor design pattern.

The visitor design pattern is often applied to describe traversal algorithms over Abstract Syntax Trees (ASTs) in imperative programming languages. It defines a *visitor*, an object with a visit method that is executed for each node in the AST, and updates the state of the visitor, and possibly the states of nodes as well. The order in which the visitor visits the nodes is explicitly under control of the programmer, which is essential to deal with the side-effectful computations that modify the state of the visitor. However, the exchange of results between traversals is error-prone.

Attribute grammars with a statically ordered attribute evaluation (Section 1.3.5) are an alternative way to describe multi-traversal algorithms. An Attribute Grammar (AG) defines attributes of nodes in the AST as functions of other attributes, and an attribute evaluator decorates the AST with the attributes in one or more traversals. The attributes form a convenient mechanism to exchange results between traversals. A strong point of AGs is that the order of evaluation is implicit. As a consequence, however, AGs discourage the use of side effects.

We present RulerCore, a language that combines attribute grammars with visitors. In RulerCore, sufficient assumptions can be made about the evaluation order to facilitate side effects. In Chapter 4 we show how to formally reason with such side effect.

A RulerCore grammar can be used in combination with several host languages. In the outline of this chapter (Section 2.1) we sketched RulerCore with the purely functional, statically typed language Haskell as host language. In this chapter, we actually show RulerCore in combination with the imperative and dynamically typed language JavaScript[1]. This chapter thus introduces the concepts that underly the subsequent chapters without a dependency on knowledge of Haskell. Also, it serves as a basis of how contents of the subsequent chapters can be mapped to other languages than Haskell.

3.1 Introduction

Algorithms for traversing tree-shaped data structures appear in many applications, especially in compilers. A lot of effort has been invested in developing proper abstractions for tree traversals, for example in the form of a tree-walking automaton (Section 1.3.3), or in a more abstract way with Attribute Grammars (AGs) [Knuth, 1968].

[1] In the outline of this chapter, we limited side effects in RulerCore to rules that determine the shape of children. Since we cannot enforce the absence of side effects in JavaScript expressions, we do not impose this restriction. Instead we present a pin-rule, which can be restricted to a visit and allows for safe use of side effect.

AGs are an attractive language for the development of compilers. We applied AGs in many small projects (to teach compiler construction [Utrecht, 2010], master projects, etc.), and several large projects, including the Utrecht Haskell Compiler [Dijkstra et al., 2009], the Helium [Heeren et al., 2003b] compiler, and the editor Proxima [Schrage and Jeuring, 2004]. AGs are an important asset in these projects. The example in Section 3.2 demonstrates some of the reasons.

Tree traversals play a role in many other fields, including end-user applications. Web applications, for example, traverse and compute properties of DOM trees. Unfortunately, the abstractions that emerge from research in compiler construction are not used to write such traversals. To use AGs, sufficient familiarity with the formalism is required, which may be an obstacle for many programmers. Also, tool support is typically absent for the programming language in question, and the AG formalism poses severe restrictions to be used effectively in these areas, such as prohibition of side effect. In this chapter, we treat the latter two challenges, which are of a technical nature.

Considering the first challenge, for imperative languages like JavaScript, a programmer either writes recursive functions, or takes a more structured approach via the visitor design pattern [Gamma et al., 1993, Palsberg and Jay, 1998, Oliveira et al., 2008]. Tool support for the visitor design pattern is available for many languages. For example, the parser generator SableCC [Gagnon and Hendren, 1998] generates visitor skeleton code for the ASTs that the parser produces. With visitors, side effects are used to carry over results computed in one visit to the next visit. In our experience, the scheduling of visits and their side effects is an error-prone process, due to the absence of the define-before-use guarantee. We elaborate on this in Section 3.2.1.

Attribute grammars offer a programming model where each node in the AST is associated with named values that are called *attributes*. An AG description contains computations that define attributes in terms of other attributes. If these definitions are noncircular, the description can be translated to a multi-visit traversal algorithm where each attribute is defined before it is used. The scheduling of the computations in implicit, which saves a programmer from writing the scheduling manually, and thus also cannot do it wrong. However, the implicit scheduling comes with a severe restriction: side effects cannot be used reliably and should not be used in attribute computations. In web applications, for example, we typically need side effects to influence the contents of a webpage. We elaborate on this in Section 3.2.2.

The main contribution of this Chapter is an extension of attribute grammars that has an explicit notion of visits, which offers a hybrid model between visitors and attribute grammars, while maintaining the best of both worlds. In fact, besides being more expressive, our extension make attribute grammars more intuitive to use.

We also address the second challenge, which is to make our approach available for many host languages. We present RulerCore, a small but powerful language for tree traversals. We managed to isolate the language-dependent part into a small subset called RulerBack, and show the translation from RulerBack to JavaScript. In later chapters, we show a translation to Haskell. With these two languages, we cover the implementation issues regarding the full spectrum of mainstream general purpose programming languages available today.

Similar to other preprocessed languages, code fragments of the host language are embedded in RulerCore to describe the computations for attributes. The embedding keeps general-purpose programming constructs out of RulerBack, and allows the programmer to express computations without having to learn a special language. In particular, RulerBack is suitable as a host language for attribute grammars.

In this chapter, we present the languages RulerCore and RulerBack. We do so by using an example based on the alignment of HTML menus. This example requires a traversal of the AST to determine the sizes of the HTML items, and another pass to compute the locations of the items. Section 3.2 presents the example in each of the above languages.

This chapter focussed on RulerBack. We introduce RulerBack in Section 3.3 and show a translation to JavaScript in In Section 3.4. In Section 3.5 we get back to RulerCore and describe the translation to RulerBack.

3.2 Example

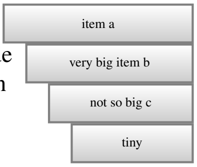
In this section, we motivate the claims of the introduction in more detail, and introduce the background information relevant for the remainder of the chapter. We take as a use case the alignment of an HTML menu in a web application using JavaScript, based on a multi-visit tree traversal over an abstract description of the menu. We first show a solution using the visitor-pattern, then a near-solution using attribute grammars, and finally two solutions using RulerCore.

3.2.1 Visitor Design Pattern

In the visitor design pattern, each node of the Abstract Syntax Tree (AST) is modelled as an object, which stores references to the subtrees, and has an *accept* method. The *accept* method takes a visitor as parameter. A visitor is an object with a *visit*-method for each type of node. The *accept* method of the AST node calls the appropriate *visit*-method on the visitor and passes the node as an argument. This *visit* method consists of statements that manipulate the state of the visitor and the AST node, and can visit a subtree by calling the *accept* method on the root of a subtree, with the visitor-object as parameter.

Figure 3.1 shows an example of a visitor that lays out HTML items as a menu in a tree-like fashion, as visualized in the upper-right corner of the figure. The menus are aligned to the right, and submenus are slightly indented. Furthermore, we desire the items to have a minimal size, but large enough to contain their contents. The variable *root* contains an abstract description of the menu as a tree of *Menu* objects (the AST). Associated with each *Menu* object is an HTML item with the same name. We interpret the menu structure to layout the HTML items. In the first visit to the menu tree, we query the widths of the corresponding HTML items. In the second visit, we adjust the positions and sizes of these items. Some information (such as indentation based on the *depth*) is computed in the first visit, and also needed in the second visit. This information is stored as additional fields in the menu objects.

The order in which the tree is visited is clearly defined by the explicit *accept*-calls in the *visit*-methods. The order of the calls ensures that the sized of the HTML items are queried before they are resized.



```
function $Menu$ $(name, children)$ {           -- constructor of a $Menu$ AST node
   $this.name \quad = name$;                    -- the name of the element to align
   $this.children = children$;                 -- an array of children menus
}
$Menu$.prototype.$accept$ = function $(visitor)$ {
   $visitor.visitMenu\ (this)$; }              -- invokes the appropriate visit method
function $Visitor$ () {                        -- constructor of a $Visitor$ object
   $this.depth \quad\quad = 0$;                -- the depth so far in the menu tree
   $this.maximum = 0$;                         -- the maximum width observed so far
   $this.count \quad\quad = 0$; }              -- the number of menus laid out so far
var $root$ =                                   -- the menu tree and corresponding html nodes
   $new\ Menu$ ("a", [                         -- <div id="a">item a</div>
      $new\ Menu$ ("b", [                      -- <div id="b">very big item b</div>
         $new\ Menu$ ("c", [])                 -- <div id="c">not so big c</div>
      , $new\ Menu$ ("d", [])                  -- <div id="d">tiny</div>
      ])
   ]);     -- <div id="anchor" onLoad="align(root,this);"></div>
function $align$ $(root, anchor)$ {            -- aligns the html nodes according to the menu tree
   var $v = new\ Visitor$ ();                  -- creates visitor with empty state
   $v.visitMenu$ = function $(menu)$ {         -- first visit method (gets $menu$ node as param)
      $menu.elem \ = document.getElementById\ (menu.name)$;
      $menu.depth = this.depth$;               -- remember $depth$ for the second visit
      $this.maximum = Math.max\ (this.maximum, this.depth * 20 + menu.elem.clientWidth)$;
      for (var $i$ in $menu.children$) {
         $this.depth = menu.depth + 1$;        -- reset $this.depth$ to one deeper than current
         $menu.children\ [i].accept\ (this)$;  -- invokes visitor on children
      }}
   $root.accept\ (v)$;                         -- invokes the first visit (on the root)
   $v.visitMenu$ = function $(menu)$ {         -- second visit method (gets $menu$ node as param)
      var $offset = menu.depth * 20$;
      $menu.elem.style.left \quad = (anchor.offsetLeft + offset)$ + "px";
      $menu.elem.style.top \quad = (anchor.offsetTop + this.count * 30)$ + "px";
      $menu.elem.style.width \ = (this.maximum - offset)$ + "px";
      $menu.elem.style.height = 30$ + "px";
      $this.count$++;                          -- inorder numbering of nodes
      for (var $i$ in $menu.children$) {       -- invokes $visitor$ on $menu$s children
         $menu.children\ [i].accept\ (this)$;  -- $count$ should not be reset in this case
      }}
   $root.accept\ (v)$; }                       -- invokes the second visit (on the root)
```

Figure 3.1: Pseudocode of dual-visit menu alignment.

However, there are a number of issues with the above solution. In the second visit, we require that a number of values are computed in the first visit. These values are stored in the state of the AST nodes during the first visit. This approach has a number of problems. It does not guarantee that the values that we store indeed those values that we need later. Furthermore, we never remove any of these values from the state, and thus retain all memory until the AST gets deallocated. This especially becomes a problem when using large ASTs in which many results are stored.

Furthermore, the order of appearance of the statements is relevant. For example, the value *this.depth* needs to be reset at the appropriate place, and requires that the assignment to *menu.depth* is done before. Similarly, the increment to *this.count* needs to be positioned carefully. These are actually separate aspects that we would like to implement in isolation. However, separate pieces of code cannot easily be composed due to side effect.

Finally, we need to explicitly write visits to children using *accept*. Some tools generate depth-first visitors, which alleviates the need to do so. However, such approaches come with restrictions. The restriction that all statements must take place before the invocations to children is an example. In Figure 3.1 we reset *this.depth* in between visits to children. To use a depth-first visitor, we would have to move this statements, which may not be immediately possible. Moreover, in the simple example that we showed, the two visits are invoked after each other at the root. In practice, for example in type checking languages with principal types, we actually invoke multiple visits on a subtree before moving on to the next subtree. This rules out depth-first visitors, and is also error-prone to write manually.

The example in Figure 3.1 can be made more complicated by allowing menus to share submenus. The menu structure then forms an acyclic directed graph instead of a tree. With such a complication, the problems mentioned above become harder to deal with.

As a sidenote, in this chapter, we treat the AST as a fixed data structure. For example, we do not consider adding menu entries on the fly. The ideas we propose can deal with the dynamic construction of proof trees (Chapter 5), and we think that this is sufficient to deal with dynamic changes to the AST as well, but leave this topic as future work.

Below, we look for a way to generate code similar to the code above, but using a description that alleviates the programmer from the aforementioned problems.

3.2.2 Attribute Grammars

Attribute grammars take care of the problems mentioned above related to visitors, but are not flexible enough to take side effects into account. Before we show the example, we first give some background information on attribute grammars, and their encoding in JavaScript.

We introduced attribute grammars in Section 1.3.1, and we use a similar syntax here with minor differences due to the JavaScript host language and to stay close to the syntax that we introduce later with respect to RulerCore. To summarize, an attribute grammar is an extension of a context-free grammar. Nonterminals are annotated with attributes. Productions specify equations between attributes. The context-free grammar specifies the structure of the AST. Each node of the AST is associated with a production, and thus also the nonterminal of the nonterminal symbol that appears as left-hand side of the production. Each child of a node corresponds to a nonterminal symbol on the right-hand side of the production.

For example, we can denote a production as well as the structure of a node in the AST using a grammar definition (explained below):

```
grammar Menus                      -- nonterminal Menus
   prod Cons hd : Menu  tl : Menus  -- production Cons, with two nonterminals
   prod Nil                         -- production Nil, empty
```

This grammar definition introduces a nonterminal *Menus* with two productions, representing a cons-list. The first production is named *Cons*. In BNF notation, it corresponds to *Menus* → *Menu Menus*. The two nonterminals *Menu* and *Menus* in the right-hand side (RHS) have explicitly been given the respective names *hd* and *tl*. Terminals only have a name (shown later in Figure 3.2).

The grammar declaration corresponds to generated JavaScript constructor functions in the host language, which can be used to construct ASTs. Each production is mapped to a constructor function that gets as parameter an object corresponding to the symbols in the RHS of the production. Each nonterminal is mapped to a constructor function that creates a base object that each of the objects corresponding to the productions inherits. Because of inheritance, we can verify at the point of construction that the AST matches the grammar:

```
function Menus () { }              -- nonterminal Menus: base class
function Menus_Cons (hd, tl) {     -- production Cons: subclass
   this.hd = hd; assert (hd instanceof Menu);
   this.tl = tl; assert (tl instanceof Menus);
}
Menus_Cons.prototype              = new Menus ();
Menus_Cons.prototype.constructor  = Menus_Cons;
function Menus_Nil () { }          -- production Nil: subclass
Menus_Nil.prototype                = new Menus ();
Menus_Nil.prototype.constructor    = Menus_Nil;
```

Cons-lists occur often in AGs. As a shortcut, the following shorthand notation may be used, which specifies that the nonterminal *Menus* is a list of *Menu* nonterminals:

```
grammar Menus : [Menu]
```

This shorthand notation has an additional benefit: the list of menus is conceptually a cons-list in the AG description but represented efficiently as a JavaScript array in the generated code. This distinction is hidden from the programmer.

The evaluation of an attribute grammar constitutes to running an evaluation algorithm on each node. The algorithm is derived from the equations of the production that is associated with the node. The algorithm describes the decoration of the node with attributes. We assume that attributes are physically represented as JavaScript properties of the AST objects. Nodes are decorated with two types of attributes: inherited attributes are computed during evaluation of the parent of that node, and synthesized attributes are computed during evaluation of the node itself.

We declare the attributes of a nonterminal using an attribute declaration:

attr *Menu* **inh** *depth* -- inherited attribute
syn *gathMax* -- synthesized attribute

These attribute names are mapped to object properties named *_inh_depth* and *_syn_gathMax*. At some point during attribute evaluation, given a participating *Menu* object *m*, the objects properties *m._inh_depth* and *m._syn_gathMax* will be defined. An inherited attribute may have the same name as a synthesized attribute: they are mapped to differently named properties. As an aside, nodes may define a number of local attributes, which can be seen as local variables.

To give a semantics to these attributes, we organize equations (rules) per production in *semantics-blocks*. We explain the following example below:

datasem *Menu* -- nonterminal *Menu*
  prod *Menu* -- production *Menu*
    $cs{:}depth = 1 + lhs{:}depth$ -- rule
    $loc{:}width = 20 * lhs{:}width$ -- rule
    $lhs{:}gathMax = Math.max\ (loc{:}width, cs{:}gathMax)$ -- rule

The full details of the nonterminal and its semantics can be found in Figure 3.2.

The left-hand side of an equation designates an attribute. The notation for attribute occurrences *nodename*:*attrname* refers to an attribute *attrname* of some node *nodename*, where *nodename* is either the name of a child, or *loc* or *lhs*. The colon ensures that attribute occurrences are district from JavaScript notation for properties. Attribute occurrences in the left-hand side of a rule refer to inherited attributes of children, but a synthesized attribute of *lhs* and a local attribute in case of *loc*. Thus, the attributes we need to define appear as left-hand side. For example, the above attribute occurrences refer to the JavaScript properties *this.cs._inh_depth*, *this._loc_width*, and *this._syn_gathMax* respectively.

Similarly, the right-hand side consists of a JavaScript expressions, with embedded attribute occurrences. In this case, we may refer to the synthesized attributes of children, or with *lhs* to the inherited attributes of the current node. The terminals of a production are available as local attributes. In production *Menu*, there is a terminal called *name*, which is available as attribute *loc*:*name*. The translation of attribute references is similar as described above. For example, the last rule expands to the JavaScript statement:

$$this.\_syn\_gathMax = Math.max\ (this.\_loc\_width, this.cs.\_syn\_gathMax);$$

Evaluation of an attribute grammar corresponds to traversing the AST one or more times, and executing rules, according to an evaluation strategy. In this chapter, we restrict ourselves to the class of well-defined attribute grammars, whose attribute dependencies can be statically proven to be acyclic [Knuth, 1968]. For these grammars, the attributes can be computed by visiting each node a bounded number of times. This corresponds precisely with typical uses of the visitor-design pattern.

```
grammar Root  prod Root  root : Menu          -- node with a child named root
grammar Menu prod Menu name  cs : Menus       -- node with a property name, and a child cs
grammar Menus : [Menu]                        -- conceptually a cons-list, physically an array

var root = new Root_Root (                    -- the Menus are physically represented
  new Menu_Menu ("a", [                       -- as an array. However, conceptually
    new Menu_Menu ("b", [                     -- we define its attributes using the
       new Menu_Menu ("c", [ ])               -- above cons-list representation.
      , new Menu_Menu ("d", [ ])])]));

attr Menu Menus inh depth finMax count        -- gathMax: width of submenu
                syn gathMax count             -- note: count is both inh and syn

function align (root, anchor) {               -- uses embedded attribute grammars
  datasem Root prod Root                      -- equations of production Root of nont Root
    root : depth    = 0                       -- initial depth
    root : count    = 0                       -- initial count
    root : finMax   = root : gathMax          -- choose gathered max as global max

  datasem Menu prod Menu                      -- production Menu of nonterm Menu
    cs : depth      = 1 + lhs : depth         -- increase depth for submenus
    cs : count      = 1 + lhs : count         -- increase count
    lhs : count     = cs : count              -- provide the updated count to the parent

    loc : elem      = document.getElementById (loc : name)
    loc : offset    = lhs : depth * 20        -- indentation
    loc : width     = loc : offset + loc : elem.clientWidth
    lhs : gathMax   = Math.max (cs : gathMax, loc : width)
    cs : finMax     = lhs : finMax            -- pass down final maximum

    loc : dummy     = (function () {          -- side-effectful statements (wrapped)
                        loc : elem.style.left   = (anchor.offsetLeft + loc : offset) + "px";
                        loc : elem.style.top    = (anchor.offsetTop + lhs : count * 30) + "px";
                        loc : elem.style.width  = (lhs : finMax − loc : offset) + "px";
                        loc : elem.style.height = 30 + "px"; }) ()   -- unwrap directly

  datasem Menus prod Cons                     -- equations of production Cons
    hd : depth      = lhs : depth             -- pass depth downwards through the menus
    tl : depth      = lhs : depth
    hd : count      = lhs : count             -- thread the count through the menus, in an
    tl : count      = hd : count              -- in-order fashion. First to the head, then to
    lhs : count     = tl : count              -- the tail, then back up to the parent.
    lhs : gathMax   = Math.max (hd : gathMax, tl : gathMax)
    hd : finMax     = lhs : finMax            -- pass global maximum downwards
    tl : finMax     = lhs : finMax

  datasem Menus prod Nil                      -- equations of production Nil
    lhs : count     = lhs : count             -- thread count through without changing it
    lhs : gathMax   = 0                       -- initial maximum

  var inhs = new Inh_Root ();                 -- contains inh attrs of the root (empty)
  eval_Root (sem_Root, root, inhs); }         -- run the attribute evaluator
```

Figure 3.2: Attribute grammar-based near-solution to menu alignment.

From a semantics-blocks (datasem-blocks in Figure 3.2), a function is generated that contains the evaluation algorithm. For example, the function *sem_Menu* is generated from the semantics of nonterminal *Menu*. Furthermore, to interface with the decorated tree in JavaScript code, a function *eval_Menu* is generated that takes the AST, the function *sem_Menu*, and an object containing values for the inherited attributes. It applies the semantic value and returns an object with the synthesized attributes:

```
var inhs = new Inh_Menu ();
inhs.depth = 0;                             -- provide inh attrs of root
syns = eval_Menu (sem_Menu, menu, inhs);    -- initiate evaluation
window.alert (syns.gathMax);                -- access syn attrs of root
```

In Figure 3.2, we show an attribute grammar version of the example that we presented earlier. It is a non-solution, for reasons explained later, but exhibits various important properties. Below, we comment on some aspects of the example.

The attribute grammar code in Figure 3.2 starts with a number of grammar definitions that describe the structure of the menu tree. We then define a number of attributes. In particular, the idea is that we gather a maximum *gathMax* (synthesized), and use its value at the root to pass down the global maximum *finMax* (inherited). Moreover, we count the menus. The inherited attribute *count* specifies the count for the current menu, and the synthesized *count* is the count incremented with the total number of children.

We define the semantics for these attributes in the function *align*. Because *root* and *anchor* are its parameters, we also have access to these in the right-hand sides of rules.

A HTML item can be laid out using statements that assign to properties of an HTML item. Since the right-hand side of an attribute equation (rule) is an expression, a sequence of statements needs to be wrapped as an expression. In JavaScript, this can be accomplished in a variety of ways. In the example, we choose to use a parameterless anonymous function for this purpose.

In the semantics of *Menus*, rules are given to compute the attributes for lists of menus using the cons-list representation. These rules follow standard patterns. The attributes *depth* and *finMax* are passed topdown. The attribute *gathMax* is computed bottom-up. The attribute *count* is threaded through the tree. In the visitor-example, the fields in the visitor combined with side effects took care of this behavior. With attribute grammars, we have to describe it explicitly. However, with copy rules (Section 1.3.12), collection rules [Magnusson et al., 2007], and a generalization called default rules (Chapter 5), we can abstract from these patterns, so that a more concise semantics of *Menus* can be given (as we see later).

The AG code has three nice properties. Firstly, the order of appearance of the rules is irrelevant. This allows the rules for *depth* and *count* to be written separately and merged automatically [Löh et al., 1998]. In the example, we give all the rules without using such composition facilities. However, for larger projects the ability to write such rules separately is important with respect to modularity.

Secondly, a nice property is the absence of invocations of visits (the *accept* calls in the visitor-example). The number of visits is totally implicit. From the dependencies between attributes in the rules, it can be determined automatically that the attribute *root*:*gathMax* (in the semantics of *Root*) must be computed in a visit before the visit where it is passed as *root*:*finMax*.

Thirdly, we check statically if there is an evaluation order of statements such that all attributes are defined before their value is accessed. The attribute declarations describe the attributes that must be defined, and those that are available. The rules describe what attributes must be available before computing an attribute, and an evaluation order is possible if the transitive closure of the dependencies is acyclic [Knuth, 1968].

Unfortunately, when the above is evaluated on-demand it is incorrect because the order of evaluation of rules is determined is not only determined by dependencies on attributes but also by the side effects that rearrange the HTML items. Since the latter effects are not present as a dependency between rules and attributes, the order of evaluation may be wrong. In fact, the root of the tree does not have any attributes defined, so when assuming a on-demand evaluation of the grammar, it is actually expected that none of the rules are evaluated. Hence, we allow the programmer to explicitly encode the dependencies imposed by side effects in the next section.

3.2.3 RulerCore

We now present a solution using RulerCore in Figure 3.3 which resembles the code in Figure 3.2. We discuss similarities and differences below.

The essential difference is that RulerCore has notation to explicitly describe visits to an AST node during attribute evaluation, and notation to associate side effects with individual visits.

Interfaces. Instead of declaring attributes for a nonterminal, we declare an *interface* for a nonterminal. An interface declaration specifies the visits of a nonterminal and attributes per visit. The following example specifies that the attributes of *Menu* are computed in two visits:

```
itf Menu                -- interface for nonterminal Menu
  visit gather          -- declaration of first visit
    syn gathMax         -- synthesized attr computed by visit
  visit layout          -- declaration second visit
    inh finMax count    -- two inherited attributes
    syn count           -- synthesized attr computed by visit
```

The order of appearance of visit declarations dictates the order of visits to AST nodes with this interface. In order to visit a node, all previous visits must have occurred. Values for inherited attributes must be provided prior to the visit. Values for synthesized attributes are only available after a visit has been performed.

Scheduling. The rules of a semantics-block are automatically scheduled over visits using an as-late-as-possible strategy (Section 3.5.2). If the rules are cyclic, the scheduling is not possible, and a static error is reported. The scheduling determines which children to visit and in what order. However, since *Root* has no attributes, there is no need to invoke any visits of

```
grammar Root   prod Root   root : Menu          -- node with a child named root
grammar Menu   prod Menu name cs : Menus        -- node with a property name, and a child cs
grammar Menus : [Menu]                          -- conceptually a cons-list, physically an array

var root = new Root_Root (                      -- the Menus are physically represented
  new Menu_Menu ("a", [                         -- as an array. However, conceptually
    new Menu_Menu ("b", [                       -- we define its attributes using the
       new Menu_Menu ("c", [ ])                 -- above cons-list representation.
      ,new Menu_Menu ("d", [ ])])]));

itf Root visit perform                          -- root node has one visit, but no attrs
itf Menu Menus                                  -- itf for nonterminals Menus (menu nodes)
  visit gather inh depth syn gathMax            -- first visit: compute maximum
  visit layout  inh finMax count syn count      -- second visit: layout the HTML items

function align (root, anchor) {                 -- uses embedded attribute grammars
  datasem Root prod Root                        -- equations of production Root of Root
    root : depth    = 0                         -- initial depth
    root : count    = 0                         -- initial count
    root : finMax   = root : gathMax            -- global max is the gathered max here
    invoke layout of root                       -- require that visit layout of root is invoked
  datasem Menu prod Menu                        -- equations scheduled to visits of Menu
    cs : depth      = 1 + lhs : depth           -- increase depth for submenus
    cs : count      = 1 + lhs : count           -- increase count
    lhs : count     = cs : count                -- provide the updated count to the parent

    loc : offset     = lhs : depth * 20         -- indentation
    loc : width      = loc : offset + loc : elem.clientWidth
    lhs : gathMax    = Math.max (cs : gathMax, loc : width)
    cs : finMax      = lhs : finMax             -- pass down final maximum

    visit gather
      pin loc : elem = document.getElementById (loc : name)
      visit layout                              -- equations for visit layout and later
        pin _ = (function () {                 -- side-effectful statements (wrapped as function)
                   loc : elem.style.left    = (anchor.offsetLeft + loc : offset) + "px";
                   loc : elem.style.top     = (anchor.offsetTop + lhs : count * 30) + "px";
                   loc : elem.style.width   = (lhs : finMax − loc : offset) + "px";
                   loc : elem.style.height = 30 + "px";
                }) ()                          -- directly call the anonymous function
  datasem Menus                                 -- standard patterns for Menus
    default depth    = function (depths) { return depths [depths.length − 1]; }
    default finMax   = function (maxs)   { return maxs [maxs.length − 1]; }
    default gathMax = function (maxs)   { return Math.max.apply (Math, maxs); }
    default count    = function (counts) { return counts [0]; }
    prod Cons                                   -- each production must be explicitly listed,
    prod Nil                                    -- even if they do not have individual rules

  var inhs = new Inh_Root_perform ();           -- contains inh attrs for the root (empty)
  eval_Root (sem_Root, root, inhs); }           -- run the attribute evaluator
```

Figure 3.3: RulerCore solution to menu alignment.

root. Therefore, we specify through an invoke-rule that visit *layout* must be invoked, which requires through attribute dependencies that also visit *gather* must be invoked, and kickstarts the evaluation.

Scheduling constraints. Rules can be constrained to visits. Rules that appear in a visit-block are constraint to that visit or a later visit. The example below illustrates the various possibilities. An attribute definition that is prefixed with the keyword **pin** is restricted to exactly the visit that it appears in, and is executed during that visit even where there are no value dependencies on the attributes that it defines:

```
datasem Menu                    -- rules for nonterminal Menu
  prod Menu                     -- rules for production Menu
    cs:count = lhs:count + 1    -- scheduled in visit gather or later
    visit gather
      pin loc:elem = ...        -- precisely in visit gather
      visit layout              -- rules for visit layout or later
        pin _       = ...       -- precisely in visit layout
        lhs:count  = cs:count   -- constrained to layout or later
```

With an underscore, we bind the value of the RHS of a rule to an anonymous attribute that we cannot refer to anymore.

A visit-block may contain rules and optionally either a nested visit-block or a nested clause-block. We use and explain clause-blocks later.

A visit-block introduces a subscope. A local attribute defined in a visit-block is not available for a rule defined in a higher scope, even if that rule is scheduled to a subscope. Attributes of children are available to higher scopes.

After all these preparations, we finally present the RulerCore solution in Figure 3.3. In this example, we express that the side effects that query the widths of the HTML items are constrained to the first visit, and that the side effects that change the locations and dimensions are constrained to the second visit.

For the *Menus*-nonterminal, we give default-rules for equality named attributes in its productions. If such an attribute (e.g. $k_i : a$) does not have an explicit definition, it is implicitly defined by the default rule. Associated with the children in a production is their order of appearance. The default-rule provides a function which receives a list (an array in JavaScript) as argument that contains the values of the attributes a of the children in the production and preserving the order of the children. Children without an attribute a do not have a value of this list. Also, the value of inherited $lhs : a$ is added to the end of the list if it exists.

Remarks. In the above example, we combined both side effects and attribute evaluation. We retain the advantages that AGs offer, such as the ease of adding attributes. As we show later, the description still permits the AG to be analyzed and the rules to be ordered.

However, we require the programmer to manually assign attributes to visits, and to constrain side-effectful rules to particular visits, which is not necessary for conventional attribute grammars. In practice, this is only a minimal amount of extra work that has as an additional advantage that it makes attribute evaluation more predictable and thus easier to understand.

```
function align (root, anchor) {                -- uses embedded attribute grammars
  var sem_Root =                               -- semantic function with itf Root
    sem prodRoot : Root                        -- equations for itf Root
      visit perform                            -- equations for the perform, the only visit
        clause Root                            -- production named Root
          child root : Menu = sem_Menu         -- introduce a child root of nonterm Menu
          root : ast    = lhs : ast            -- use lhs : ast as AST
          root : depth  = 0                    -- initial depth
          root : count  = 0                    -- initial count
          root : finMax = root : gathMax       -- global max is the gathered max of here
          invoke layout of root                -- demand invocation layout of root
  var sem_Menu =                               -- semantic function with itf Menu
    sem prodMenu : Menu                        -- equations for itf Menu
      visit gather                             -- equations for first visit
        clause Menu                            -- production named Menu
          child cs : Menus = sem_Menus         -- introduce a child cs of nonterm Menus
          cs : ast        = lhs : ast.cs       -- pass submenus as AST for cs
          cs : depth      = 1 + lhs : depth    -- increase depth for submenus
          pin loc : elem  = document.getElementById (loc.name)
          loc : offset    = lhs : depth * 20   -- indentation
          loc : width     = loc : offset + loc : elem.clientWidth
          lhs : gathMax   = Math.max (cs : gathMax, loc : width)
          cs : finMax     = lhs : finMax       -- pass down global maximum
          visit layout                         -- equations for visit layout
            clause Menu'                       -- subproduction named Menu'
              cs : count  = 1 + lhs : count    -- increase count
              lhs : count = cs : count         -- provide the updated count to the parent
              pin _       = (function () {     -- side-effectful statements
                    loc : elem.style.left   = (anchor.offsetLeft + loc : offset) + "px";
                    loc : elem.style.top    = (anchor.offsetTop + lhs : count * 30) + "px";
                    loc : elem.style.width  = (lhs : finMax - loc : offset) + "px";
                    loc : elem.style.height = 30 + "px";
                        }) ()                  -- directly call the anonymous function
  ...  -- See Figure 3.5
  var inhs     = new Inh_Root_perform ();      -- contains inh attrs for the root
  inhs.ast     = root                          -- AST as inherited attribute
  eval_Root_perform (sem_Root, inhs);          -- run the attribute evaluator
}
```

Figure 3.4: Desugared RulerCore solution to menu alignment (part 1).

3.2.4 Desugared RulerCore

In Figure 3.4 (explained below), we give a different *desugared* of Figure 3.3. Both versions are valid RulerCore programs. This desugared version only uses a subset of RulerCore, which we call *RulerBack*. This representation is more verbose, but more suitable for code generation.

RulerBack generalizes over higher-order [Vogt et al., 1989] and conditional [Boyland, 1996] attribute grammars. In the next section, we introduce RulerBack. The example in Figure 3.4 serves as preparation. In Figure 3.4, we omit the grammar definitions, interface declaration, and *root* variable, which are equal to those in the first half of Figure 3.3.

In an attribute grammar, there is a fixed association between a node in the AST and a production and a fixed association between a production and a collection of rules. The code to execute for a node in the AST is derived from the associated collection of rules. RulerBack *virtualizes* productions: we define grammars that describe traversals instead of data structures (Section 1.3.12). The rules of a RulerBack production are organized in *clauses* (introduced below), and rules can programmatically determine which clauses to evaluate.

The above functionality allows us to define a single production per nonterminal. The nonterminal has an inherited attribute *ast* which contains the AST as an inspectable value. Note that in RulerBack the representation of cons-lists using arrays becomes visible whereas this is hidden in the RulerCore example. In the translation from RulerCore to RulerBack additional RulerBack rules are generated to treat the explicit array representation. We show this in the example.

Semantics blocks, which are of the form **sem** $P : N \ldots$, introduce a production P of nonterminal N. The visits and attributes of N are declared separately with an *interface declaration*. Additionally, the code generated from a sem-block is a constructor-function that produces an AST node with attributes as described by N. The AST is provided explicitly as the inherited attribute *ast*.

In Figure 3.4, we start with a definition of the semantics for the root. The interface *Root* declares one visit, and we give rules for that visit in a visit-block. RulerBack provides clauses as a means to generalize over productions. Each clause provides a way to compute the attribute values of a visit. Moreover, a clause may specify constraints. Clauses are executed in the order of appearance. A clause is selected if its constraints are satisfied. Conventional productions, which specify a constraint on the node of the tree, can thus be represented with clauses.

Clauses and visit-blocks may contain rules. Rules given for a visit are in scope of all clauses declared for that visit. Rules for a clause are only visible in that clause. We introduce child-rules. A child-rule introduces child. In the example, we introduce a child *root*, with interface *Menu*, and the semantics defined by the JavaScript value *sem_Menu*. This is an example of a higher-order child (Section 1.3.7) and is used to virtualize the AST. Unlike in later chapters, in this chapter the virtual AST is isomorphic to the actual AST. Also, we assume that a visit v to child x is only possible if there exists an invoke-rule for it.

The left-hand sides of an evaluation-rule may be a pattern. This is either an attribute reference, an underscore, or a constant. Evaluation of such a rule fails when its execution throws an exception or the left-hand side is a value that is not equal to the value computed for

```
function align (root, anchor) {                         -- uses embedded attribute grammars
  ...                                                   -- See Figure 3.4

  var sem_Menus =                                       -- semantic function, also itf Menu
    sem prodMenus : Menu                                -- equations for itf Menu
      visit gather                                      -- equations for visit gather
        default depth    = function (depths) { return depths [depths.length − 1]; }
        default finMax   = function (maxs)   { return maxs [maxs.length − 1]; }
        default gathMax  = function (maxs)   { return Math.max.apply (Math, maxs); }
        default count    = function (counts) { return counts [0]; }

        clause Cons                                     -- production Cons as clause
          match true = lhs : ast.length ⩾ 1             -- clause matches if array has an element

          child hd : Menu = sem_Menu                    -- introduce child hd using sem_Menu
          hd.ast          = lhs : ast [0]               -- head of the array

          child tl : Menu = sem_Menus                   -- introduce child tl using sem_Menus
          tl.ast          = lhs : ast.slice (1)         -- tail of the array
        clause Nil                                      -- production Nil (matches always)

  var inhs = new Inh_Root_perform ();                   -- contains inh attrs for the root
  inhs.ast = root                                       -- AST as inherited attribute
  eval_Root_perform (sem_Root, inhs); }                 -- run the attribute evaluator
```

Figure 3.5: Desugared RulerCore solution to menu alignment (part 2).

the right-hand side. Such a failing rule causes an exceptional termination of the evaluation, unless the evaluation-rule is prefixed with the match-keyword, and the rule does not throw an exception other than a special fail-exception. In this case, we say that the clause *fails*. Thus, the match-rules allow us to distinguish clauses *Cons* and *Nil* of *ntMenus* by matching on the length of the list.

Invoke rules and visit-blocks may be annotated with strategies of various kinds. Chapter 2 describes these strategies: in this chapter the strategies **partial** and **total**, which describe backtracking behavior, appear.

During attribute evaluation, the clauses of a visit are evaluated in the order of appearance. When evaluation for a clause fails, the evaluation *backtracks* to the next clause[2]. A backtrack does not revert potential side effects of the results that are evaluated so far. If the last clause fails, the default behavior is that the evaluation fails exceptionally. However, if both the visit itself and the invoke-rule of the parent are annotated with the strategy **partial**, then the invoke-rule of the parent fails and causes backtracking in the parent. By default, a total strategy is assumed.

Unspecified visit-blocks are implicitly defined as an empty visit-block. A visit-block without clauses implicitly has a single clause. This clause matches always unless match-rules are present. Therefore, we neither have to specify the visit-block *layout* nor clauses for it in the

[2] Chapter 7 describes a different strategy where clauses are simultaneously tried.

$e ::= \mathcal{J}\ [\overline{b}]$	-- embedded RulerBack blocks b in JavaScript code $\mathcal{J}$
$b ::= i \mid s \mid o$	-- RulerBack blocks
$i\ ::= \textbf{itf}\ I\ v$	-- interface decl, with visit sequence v
$v ::= \textbf{visit}\ x\ \textbf{inh}\ \overline{x_1}\ \textbf{syn}\ \overline{x_2}\ z\ v$	-- visit decl, with atributes x_1 and x_2 and strategy z
$\mid\ \Box$	-- terminator visit (optional in the notation)
$s ::= \textbf{sem}\ x : I\ t$	-- semantics expr, defines production x
$t\ ::= \textbf{visit}\ x\ \overline{r}\ \overline{k}$	-- visit-block, with rules r and clauses $\overline{k}$
$\mid\ \Box$	-- no visit (serves as terminator)
$k ::= \textbf{clause}\ x\ \overline{r}\ t$	-- clause definition, with next visit t
$r ::= p = e$	-- assert-rule, evaluates e, binds to pattern p
$\mid\ \textbf{pin}\ p = e$	-- pinned assert rule (bound to visit it occurs in)
$\mid\ \textbf{match}\ p = e$	-- match-rule, backtracking variant
$\mid\ \textbf{invoke}\ x\ \textbf{of}\ \overline{c}\ z$	-- invoke-rule, invokes visit x on $\overline{c}$
$\mid\ \textbf{child}\ c : I = e$	-- child-rule, introduces a child c of itf I
$z ::= \textbf{partial} \mid \textbf{total}$	-- behavior in case of rule failure
$o ::= c : x$	-- attribute reference in some embedded code
$p ::= c : x$	-- attribute reference in pattern
$\mid\ \_$	-- wildcard
$\mid\ K$	-- constant K

$x, c\ p, e$ -- identifiers, child identifiers, patterns, expressions respectively

$\Gamma, \Sigma ::= \varepsilon$	-- attr+child environment (used in semantics)
$\mid\ \Gamma, \circ$	-- new scope
$\mid\ \Gamma, \textbf{inh}\ c : x$	-- inh attr $c : x$
$\mid\ \Gamma, \textbf{syn}\ c : x$	-- syn attr $c : x$
$\mid\ \Gamma, c : I\ v$	-- child c with available visit sequence v
$\Phi ::= \varepsilon$	-- interface environment (used in semantics)
$\mid\ \Phi, I\ v$	-- itf I with visit decls v

Figure 3.6: Syntax of RulerBack

```
itf S visit v₁ inh l  syn ∅ total      -- decompose array l down
    visit v₂ inh ∅ syn s total         -- compute sum s up
    □                                  -- end of visit decls
var sumArr = sem sum : S
  visit v₁ ∅                           -- first visit
    clause sumNil                      -- when list is empty
      match 0 = lhs : l.length         -- match empty l
      visit v₂ ∅                       -- second visit
        clause sumNil2                 -- single clause
          lhs : s = 0                  -- empty list, zero sum
          □                            -- end of visit blocks
    clause sumCons                     -- when list non-empty
      loc : x  = lhs : l [0]           -- head of the list
      loc : xs = lhs : l.slice (1)     -- tail of the list
      child tl : S = sumArr            -- recursive call
      tl : l = loc : xs                -- l param of call
      invoke v₁ of tl total            -- invoke on child
      visit v₂ ∅                       -- second visit
        clause sumCons2                -- single clause
          invoke v₂ of tl total        -- invoke on child
          lhs : s = loc : x + tl : s   -- sum of head and tail
          □                            -- end of visit blocks
```

Figure 3.7: Example of RulerBack syntax: summing an array of integers.

semantics of *ntMenus*. Also, because of the automatic ordering of rules, many of the rules defined in visit *layout* of *ntMenu*, could also be defined one level higher, in visit *gather*.

Note that this representation is more general than conventional attribute grammars, and that an attribute grammar can easily be mapped to this representation, as shown by the difference between Figure 3.3 and Figure 3.4.

3.3 Static Semantics of RulerBack

In this section, we introduce RulerBack, a small subset of RulerCore. It serves as an intermediate language for RulerCore. Figure 3.6 shows the syntax of RulerBack. A RulerBack program *e* is a JavaScript program $\mathcal{J}$, with embedded RulerBack blocks *b*. A block *b* is either an interface declaration, semantics-block, or attribute reference. The syntax of visits in interface declarations and semantics-blocks use a cons-list representation which is convenient for the specification of translation schemes later. We explain the individual forms of syntax in more detail below.

There are some essential differences with respect to RulerCore that we gradually introduced in the previous section. The order of appearance of rules defines the evaluation order, and each invocation of a visit must explicitly be stated through an invoke rule. Grammar blocks can be desugared and are optional in RulerBack. Instead, with clauses and (match) rules, we provide a general mechanism to traverse arbitrary JavaScript data structures.

The embedded blocks may occur anywhere in a JavaScript program. The programmer is required to position semantics blocks and attribute references at expression positions in the host language, and interface declarations at statement positions. It is the responsibility of the programmer to handle the scoping of embedded blocks.

Figure 3.7 shows a RulerBack program that computes the sum of an array of integers in two visits. This simple example can also be formulated as a single visit. However, it serves here as a short example of a dual-visit program. The first visit has two clauses: a clause *sumNil* when the array is empty, and *sumCons* when there is at least one element. In the second visit, the actual sum is computed, using the rules that depend on which clause is chosen in the first visit.

A semantics-block introduces a visitor-object with an interface I. The interface dictates what visits can be made to the object, and what the inputs (inherited attributes) and outputs are (synthesized attributes). The outputs for a visit are produced by executing rules. We write these rules down in a tree of clauses and visits, as illustrated by the indentation in Figure 3.7 and the state diagram in Figure 3.8.

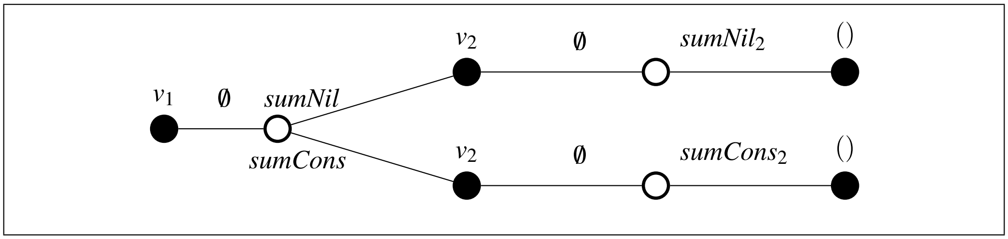


Figure 3.8: States of nodes with semantics *sum*.

The black nodes represent the state of the AST-node prior to a visit and the white nodes indicate a branch point. Upon creation, an AST node is in the state represented by the root node. With each edge, alternatively the rules of a visit or the rules of a clause are associated. With each visit, an AST node changes state to a next black node by executing the rules on the path to such a node. Execution of all of the rules must succeed. At a branch-point, rules on edges of clauses are tried in order of appearance. Results produced by executing rules are in scope of rules further along the path.

There are four types of rules in RulerCore.

- **match** $p = e$ -- match-rule
  match $loc{:}x = 3$ -- example that succeeds
  match *true* $=$ *false* -- example that fails

  The pattern p must match the value of the right hand side. If the evaluation of e results in an exception, or the match fails, a backtrack is made to the next clause. If p

$$\begin{array}{llll l}
\Gamma\vdash s & v\;;\;\Gamma\vdash t & \Sigma\;;\;\Gamma_0\vdash r\;:\;\Gamma_1 & \Gamma\vdash e & \text{-- signatures of the relations}\\
\Gamma\vdash o & v\;;\;\overline{x}\;;\;\Gamma\vdash c & \Sigma\;;\;\Gamma_0\vdash p\;:\;\Gamma_1 & & \text{-- used by judgments below}
\end{array}$$

$$\frac{\begin{array}{c}x\ \mathit{unique}\\ I\,v\in\Phi \qquad v\;;\;\Gamma,\circ\vdash t\end{array}}{\Gamma\vdash\mathbf{sem}\,x\!:\!I\,t}\ \textsc{sem} \qquad\qquad \square\;;\;\Gamma\vdash\square\ \textsc{end}$$

$$\frac{\Gamma_0\cup\mathit{avail}\,(\mathbf{visit}\,x\,\overline{r}\,\overline{k})\;;\;\Gamma_0\cup\{\mathbf{inh}\,\mathit{lhs}\!:\!a\mid a\in\overline{i}\}\vdash\overline{r}\;:\;\Gamma_1 \qquad \overline{v\;;\;\overline{s}\;;\;\Gamma_1\vdash k}}{\mathbf{visit}\,x\,\mathbf{inh}\,\overline{i}\,\mathbf{syn}\,\overline{s}\,v\;;\;\Gamma_0\vdash\mathbf{visit}\,x\,\overline{r}\,\overline{k}}\ \textsc{visit}$$

$$\frac{\begin{array}{c}x\ \mathit{unique}\\ \Gamma_0\cup\mathit{avail}\,(\mathbf{clause}\,x\,\overline{r}\,\overline{k})\;;\;\Gamma_0\vdash\overline{r}\;:\;\Gamma_1 \qquad v\;;\;\Gamma_1\vdash t \qquad \{\mathbf{syn}\,\mathit{lhs}\!:\!a\mid a\in\overline{s}\}\subseteq\Gamma_1\end{array}}{v\;;\;\overline{s}\;;\;\Gamma_0\vdash\mathbf{clause}\,x\,\overline{r}\,t}\ \textsc{clause}$$

$$\frac{\Sigma\;;\;\Gamma_0\vdash p\;:\;\Gamma_1 \qquad \Gamma_0\vdash e}{\Sigma\;;\;\Gamma_0\vdash\mathbf{pin}^{?}\,p=e\;:\;\Gamma_1}\ \textsc{assert} \qquad\qquad \frac{\Sigma\;;\;\Gamma_0\vdash p\;:\;\Gamma_1 \qquad \Gamma_0\vdash e}{\Sigma\;;\;\Gamma_0\vdash\mathbf{match}\,p=e\;:\;\Gamma_1}\ \textsc{match}$$

$$\frac{\begin{array}{c}\Phi\,(I_c)=\overline{v} \qquad \mathbf{visit}\,x\,\mathbf{inh}\,\overline{i}\,\mathbf{syn}\,\overline{s}\,z_2\in\overline{v} \qquad z_1\sqsubseteq z_2 \qquad c\!:\!I_c\,\overline{w}\in\Gamma_0 \qquad \mathit{next}\,\overline{w}\,\overline{v}=x\\ \{\mathbf{inh}\,c\!:\!a\mid a\in\overline{i}\}\subseteq\Gamma_0 \qquad \Gamma_1=\Gamma_0\cup\{\mathbf{syn}\,c\!:\!a\mid a\in\overline{s}\}\cup\{c\!:\!I_c\,(\overline{w},\mathbf{visit}\,x\,\mathbf{inh}\,\overline{i}\,\mathbf{syn}\,\overline{s})\}\end{array}}{\Sigma\;;\;\Gamma_0\vdash\mathbf{invoke}\,x\,\mathbf{of}\,c\,z_1\;:\;\Gamma_1}\ \textsc{invoke}$$

$$\frac{\Gamma_0\vdash e \qquad \Gamma_1=\Gamma_0\cup\{c\!:\!I\,\emptyset\}}{\Sigma\;;\;\Gamma_0\vdash\mathbf{child}\,c\!:\!I=e\;:\;\Gamma_1}\ \textsc{child} \qquad \frac{\mathbf{inh}\,\mathit{lhs}\!:\!a\in\Gamma}{\Gamma\vdash\mathit{lhs}\!:\!a}\ \textsc{occ.lhs} \qquad \frac{\mathbf{syn}\,c\!:\!a\in\Gamma}{\Gamma\vdash c\!:\!a}\ \textsc{occ.child}$$

$$\frac{\mathbf{syn}\,\mathit{lhs}\!:\!a\in\Sigma}{\Sigma\;;\;\Gamma_0\vdash\mathit{lhs}\!:\!a\;:\;\Gamma_0,\mathbf{syn}\,\mathit{lhs}\!:\!a}\ \textsc{pat.lhs} \qquad\qquad \Sigma\;;\;\Gamma_0\vdash\mathit{loc}\!:\!a\;:\;\Gamma_0,\mathbf{syn}\,\mathit{loc}\!:\!a\ \textsc{pat.loc}$$

$$\frac{\mathbf{inh}\,c\!:\!a\in\Sigma}{\Sigma\;;\;\Gamma_0\vdash c\!:\!a\;:\;\Gamma_0,\mathbf{inh}\,c\!:\!a}\ \textsc{pat.child} \qquad\qquad \Sigma\;;\;\Gamma\vdash K\;:\;\Gamma\ \textsc{const} \qquad\qquad \Sigma\;;\;\Gamma\vdash\_\;:\;\Gamma\ \textsc{any}$$

$$\begin{array}{ll}
\mathit{avail}\,(\mathbf{visit}\,x\,\overline{r}\,\overline{k}) & =\mathit{avail}_{\cup}\,(\overline{r})\cup\mathit{avail}_{\cap}\,(\overline{k})\\
 & \quad\cup\{\mathbf{syn}\,\mathit{lhs}\!:\!b\mid\mathbf{visit}\,x\,\mathbf{inh}\,\overline{a}\,\mathbf{syn}\,\overline{b}\in\Phi\,(I_x)\}\\
\mathit{avail}\,(\mathbf{clause}\,x\,\overline{r}\,t) & =\mathit{avail}_{\cup}\,(\overline{r})\cup\mathit{avail}\,(t)\\
\mathit{avail}\,(p=e) & =\emptyset\\
\mathit{avail}\,(\mathbf{match}\,p=e) & =\emptyset\\
\mathit{avail}\,(\mathbf{invoke}\,x\,\mathbf{of}\,c) & =\{\mathbf{inh}\,c\!:\!a\mid a\in\overline{a},\mathbf{visit}\,x\,\mathbf{inh}\,\overline{a}\,\mathbf{syn}\,\overline{b}\in\Phi\,(I_c)\}\\
\mathit{avail}\,(\mathbf{child}\,c\!:\!I=e) & =\{c\!:\!I\,(\Phi\,I)\}
\end{array}$$

Figure 3.9: Static semantics of RulerBack

represents an attribute, the attribute gets defined.

- **pin**$^?$ $p = e$ -- eval-rule (optionally pinned)

Similar to the above, except that the match is expected to succeed. If not, the evaluation itself aborts with an exception. For that reason, we also call these rules assert-rules. Note that these are the conventional rules of attribute grammars.

- **child** $c : I = e$ -- child-rule
  child $root : Menu = ntMenu$ -- example that introduces a *Menu* child

Evaluation of the rule above creates a child c, visitable according to the interface I, and created by executing the constructor function e.

- **invoke** x **of** c z -- invoke rule

Executes visit x of child c. The inherited attributes of x must be defined, and all prior visits to c must have been performed. The invocation fails if no clause matches and the strategy z is **partial**. Otherwise, the evaluation aborts exceptionally. If successful, the synthesized attributes of x become available. If there is an invoke with a partial-annotation, then the visit of the corresponding interface must also have a partial-annotation.

Figure 3.9 shows RulerBack's static semantics for sem-blocks. We omitted the rules that state the uniqueness of interfaces and attributes of interfaces. A RulerBack program that satisfies these conditions never crashes due to an undefined attribute, invalid rule order, or forgotten invocation to a child. The dynamic or static type checking we leave as responsibility of the host language.

We briefly consider some aspects of these rules. Three environments play an important role. The environment Φ contains for each interface the sequence of visits. The environment Γ represents the children and attributes defined so far for one node (to test for missing and duplicated definitions). The environment Σ represents the attributes that are allowed to be defined (to test for definitions of unknown attributes). As additional constraint on environments, we consider it a static error when there is a duplicate attribute in the environment within two scope markers.

Visit-blocks must be specified in the order as declared on the interface, and none may be omitted. The relation for visits t gets a sequence of pending visits $\overline{v}$ as declared in the interface. In rule VISIT, we verify that the name of the visit matches the expected visit in the head of $\overline{v}$. The next visit must match the head of the tail of this list, until in the end $\overline{v}$ is empty. We also add the inherited attributes of the visits to the environment.

The function *avail* defines which attributes may be defined. Higher-up in the visit-clauses-tree, we may only define those attributes that are common to all lower clauses. In rules PAT.LHS and PAT.CHILD, we verify that we are indeed defining an attribute belonging to a certain child. The avail-function either generalizes over lists using intersection or union.

Rule SEM forms the root of derivation trees. Since semantics blocks can occur nested, Γ contains potential bindings in scope from encapsulating semantics blocks. A new scope entry is added to Γ. The rule matches the visits blocks t against the declared visits v. Rule END matches with the end of a sequence of visits blocks, and requires also to have reached the end of the declared visits. Rule VISIT requires that the clauses and rules are well-formed. From the rules $\overline{r}$ it obtains an environment Γ_1 with additional bindings that are available to the clauses. Both the rules and the clauses may refer to the inherited attributes of the visit. Each clause must define the set $\overline{s}$ of synthesized attributes of the visit. We assume that the type rule for a rule r is lifted to a list of rules $\overline{r}$ by chaining the environment Γ.

Rule CLAUSE verifies that the rules it contains and the next visit are well-formed. Furthermore, it verifies that the synthesized attributes of the visit are defined. The type rules ASSERT, MATCH, INVOKE, and CHILD correspond to RulerBack rules. The ASSERT and MATCH rules verify that the pattern and expression are well-formed. The CHILD rule adds the child in an empty state to the environment.

Rule INVOKE verifies that visit x is indeed the next visit in the expected sequence of visits $\overline{v}$, given the previous invocations $\overline{w}$. It furthermore verifies that the inherited attributes for the visit of c are defined, and adds the synthesized attributes to the environment. Finally, the strategy annotation of the invoke-rule must be greater than the strategy annotation of the visit. If the visit is declared as total, then as sanity-check, an invoke may not have **partial** as strategy.

3.4 Translation of RulerBack to JavaScript

In this section, we describe how to translate RulerBack programs to JavaScript. We translate each semantics-block to a coroutine, which we implement as one-shot continuations. Each call to the coroutine represents a visit. The parameters of the coroutine are the inherited attributes of the visit. The result of the call is an object containing values for the synthesized attributes, and the continuation to call for the visit.

As an example, we show in Figure 3.10 the translation of the example in the previous section. To deal with backtracking, we use the exception mechanism, and throw an exception to switch to the next clause. Note that this does not rollback any side effects that the partial execution of the rules may have caused. To be able to do so, we can run the rules in a software transaction [Heidegger et al., 2010], which are nowadays supported by many programming languages. Alternatively, when the side effects matter, the programmer can schedule the rule to an earlier or later visit, such that it is not influenced by backtracking.

To deal with continuations, we use closures. The function to be used for the next visit is constructed in the previous visit. This function has access to all the results computed in the previous visit. Furthermore, we store values for attributes in local variables. Those values that are not needed anymore are automatically cleaned up by the garbage collector.

Figure 3.11 shows the general translation scheme, and the naming scheme for attributes. In particular, for each visit, we generate a closure that takes values for inherited attributes as parameter. Clauses are dealt with through exception handling. When a clause successfully executed all statements, it returns an object containing values for synthesized attributes, as

```
var sumArr = function () {                      -- semantic function
  function nt_sum (_inps) {                     -- visit v1
    var lhsIl = _inps.l;                        -- extract lhs:l
    try {                                       -- try clause sumNil
      if (lhsIl.length ! = 0) throw eEval;      -- if lhs:l is empty
      var _res = new Object ();                 -- produce results of v1
      _res._next = function (_inps) {           -- cont. for visit v2
          var lhsSs  = 0;                       -- lhs:s rule
          var _res   = new Object ();           -- produce results of v2
          _res._next = null;                    -- no next visit
          _res.s     = lhsSs;                   -- store lhs:s
          return _res;                          -- return result of v2
        };
      return _res;                              -- return result of v1
    } catch (err) {                             -- try clause sumCons
        var locLx  = lhsIl [0];                 -- loc:x rule
        var locLxs = lhsIl.slice (1);           -- loc:xs rule
        var vis_tl  = sumArr ();                -- creation of child tl
        tlIl = locLxs;                          -- tl:l rule
        var _args  = new Object ();             -- inputs for v1 of tl
        _args.l    = tlIl;                      -- store tl:l
        var _res   = vis_tl (_args);            -- invoke v1 of tl
        var vis_tl  = _res._next;               -- extract results
        var _res = new Object ();               -- produce results of v1
        _res._next = function (_inps) {         -- cont. for visit v2
          var _args  = new Object ();           -- inputs for v2 of tl
          var _res   = vis_tl (_args);          -- invoke v2 of tl
          var tlSs   = _res.s;                  -- extract tl:s result
          var lhsSs  = locLx + tlSs;            -- compute lhs:s
          var _res   = new Object ();           -- produce results of v1
          _res._next = null;                    -- no next visit
          _res.s     = lhsSs;                   -- store lhs:s
          return _res;                          -- return result of v2
          };
      return _res;                              -- return result of v1
    }};return nt_sum; };                        -- return visitor function
```

Figure 3.10: Example translation

well as the continuation function for the next visit.

The above translation is relatively straightforward. In practice, the selection of a clause is functionally dependent on the value of an inherited attribute, or a local attribute computed in a previous visit. In those cases, the selection of clauses can be implemented more efficiently using conventional branching mechanisms. Also, instead of using the exception mechanism to implement backtracking, we can use code duplication, which results in more efficient code. Chapter 9 shows such a translation scheme.

We verified that the above implementation runs in time linear in the size of the tree when we use a version of the *slice* operation that does not make a copy of the array. With a throughput of about hundred array elements per microsecond, and about a thousand per microsecond with the exception handling replaced by conventional branching, this is still about one or two orders of magnitude slower than using a hand-written loop. In our experience, however, the traversal performance is rarely an issue. In general, the asymptotic complexity of the traversal is linear in the size of the tree, and the actual time taken by traversing the trees is insignificant compared to the work performed by the right-hand sides of the rules in a real application.

3.5 Translation of RulerCore to RulerBack

In Section 3.2.4, we showed in an example how a RulerCore program can be encoded using only syntax of RulerBack. We omit the data-type driven translation from a **datasem** into a **sem**, nor the translation of default-rules. Instead, in this section we assume that RulerCore consists of those programs that after insertion of invoke-rules and reordering of rules are a valid RulerBack program.

3.5.1 Implicit Invocations

In RulerCore, invoke-rules may be omitted. From a RulerBack program, we derive a number of implicit invocations. We first determine the attributes that are *needed*. From these we determine the maximum needed visit, and thus the sequence of visits that is needed. An invoke-rule needs to be inserted if there is no invoke-rule for any of these visits yet. We start the insertion-process at the root of the tree, and check at each level downwards which invokes need to be inserted. With this process, we insert the invoke-rules at the lowest point, while still being in scope of all rules that need it. Automatic rule ordering then positions the invokes at their appropriate places.

A synthesized attribute a of child c is needed if there exists a rule which has the attribute reference $c\!:\!a$ in its right-hand side. The needed attributes may differ per clause and visit, which we define in a similar way as *avail* in Section 3.3:

$$\begin{aligned}
&need\ (\textbf{visit}\ x\ \overline{r}\ \overline{k}) &&= need_{\cup}\ \overline{r} \cup need_{\cap}\ \overline{k}\\
&need\ (\textbf{clause}\ x\ \overline{r}\ t) &&= need_{\cup}\ \overline{r} \cup need\ t\\
&need\ (p = e) &&= need\ e
\end{aligned}$$

Note that *need* generalizes to lists by using either intersection or union, which we denoted explicitly with $need_{\cup}$ and $need_{\cap}$.

$$
\begin{array}{lcl}
\llbracket \mathbf{sem}\ x\!:\!I\ t \rrbracket & \leadsto & \mathbf{function}\ ()\ \{\ \mathbf{var}\ \llbracket nt\ x \rrbracket = \llbracket t \rrbracket_I; \mathbf{return}\ \llbracket nt\ x \rrbracket;\ \} \\
\llbracket c\!:\!x \rrbracket & \leadsto & \llbracket inp\ c\ x \rrbracket \\
\llbracket \varepsilon \rrbracket_I & \leadsto & null \\
\llbracket \mathbf{visit}\ x\ \overline{r}\ \overline{k}\ z \rrbracket_I & \leadsto & \mathbf{function}\ (\_inps)\ \{ \\
 & & \quad \llbracket inp\ lhs\ (inhs\ I\ x) \rrbracket = \_inps.\llbracket inhs\ I\ x \rrbracket; \\
 & & \quad \llbracket \overline{r} \rrbracket; \llbracket \overline{k} \rrbracket_{I,z,syns\ I\ x};\ \} \\
\llbracket [\,] \rrbracket_{I,\mathbf{partial},\overline{s}} & \leadsto & \mathbf{throw}\ eEval; \\
\llbracket [\,] \rrbracket_{I,\mathbf{total},\overline{s}} & \leadsto & \mathbf{throw}\ eAbort; \\
\llbracket k\!:\!\overline{k} \rrbracket_{I,z,\overline{s}} & \leadsto & \mathbf{try}\ \{\ \llbracket k \rrbracket_{I,\overline{s}};\ \} \\
 & & \mathbf{catch}\ (err)\ \{ \\
 & & \quad \mathbf{if}\ (err == eEval)\ \{\ \llbracket \overline{k} \rrbracket_{I,z,\overline{s}};\ \} \\
 & & \quad \mathbf{else\ throw}\ err;\ \} \\
\llbracket \mathbf{clause}\ x\ \overline{r}\ t \rrbracket_{I,\overline{s}} & \leadsto & \llbracket \overline{r} \rrbracket; \\
 & & \mathbf{var}\ \_outs = new\ Object\ (); \\
 & & \_outs.\_next = \llbracket t \rrbracket_I; \\
 & & \_outs.\overline{s} = \llbracket out\ lhs\ \overline{s} \rrbracket; \\
 & & \mathbf{return}\ \_outs; \\
\llbracket \mathbf{pin}^?\ p = e \rrbracket & \leadsto & \llbracket \mathbf{var}\ \_res = \llbracket e \rrbracket \rrbracket_{\mathbf{total}}; \\
 & & \llbracket p \rrbracket_{eAbort} \\
\llbracket \mathbf{match}\ p = e \rrbracket & \leadsto & \mathbf{var}\ \_res = \llbracket e \rrbracket; \llbracket p \rrbracket_{eEval}; \\
\llbracket \mathbf{child}\ c\!:\!I = e \rrbracket & \leadsto & \mathbf{var}\ \llbracket vis\ c \rrbracket = (\llbracket e \rrbracket)\ (); \\
\llbracket \mathbf{invoke}\ x\ \mathbf{of}\ c\ z \rrbracket & \leadsto & \mathbf{var}\ \_args = new\ Object\ (); \\
 & & \_args.\llbracket inhs\ I_c\ x \rrbracket = \llbracket out\ c\ (inhs\ I_c\ x) \rrbracket; \\
 & & \llbracket \mathbf{var}\ \_res = \llbracket vis\ c \rrbracket\ (\_args) \rrbracket_z; \\
 & & \mathbf{var}\ \llbracket inp\ c\ (syns\ I_c\ x) \rrbracket = \_res.\llbracket syns\ I_c\ x \rrbracket; \\
 & & \mathbf{var}\ \llbracket vis\ c \rrbracket = \_res.\_next; \\
\llbracket e \rrbracket_{\mathbf{partial}} & \leadsto & e \\
\llbracket e \rrbracket_{\mathbf{total}} & \leadsto & \mathbf{try}\ \{\ e;\ \}\ \mathbf{catch}\ (err)\ \{ \\
 & & \quad \mathbf{if}\ (err == eEval)\ \mathbf{throw}\ eAbort; \mathbf{else\ throw}\ err;\ \} \\
\llbracket c\!:\!a \rrbracket_e & \leadsto & \mathbf{var}\ \llbracket out\ c\ a \rrbracket = \_res; \\
\llbracket \_ \rrbracket_e & \leadsto & ; \\
\llbracket k \rrbracket_e & \leadsto & \mathbf{if}\ (\_res \,!\!= k)\ \mathbf{throw}\ e;
\end{array}
$$

$$
\begin{array}{ll}
out\ \texttt{"loc"}\ x = \texttt{"locL"}\ x & inp\ \texttt{"loc"}\ x = \texttt{"locI"}\ x \\
out\ \texttt{"lhs"}\ x = \texttt{"lhsS"}\ x & inp\ \texttt{"lhs"}\ x = \texttt{"lhsI"}\ x \\
out\ c\ \ \ \ \ \ \ x = c\ \texttt{"I"}\ x & inp\ c\ \ \ \ \ \ \ x = c\ \texttt{"S"}\ x \\
vis\ c = \texttt{"vis\_"}\ c & nt\ x = \texttt{"nt\_"}\ x \\
syns\ I\ x \qquad inhs\ I\ x & \text{-- respectively, inh and syn attrs of } x \text{ of } I
\end{array}
$$

Figure 3.11: Denotational semantics of RulerBack

In our actual implementation, we defined the function *need* in a slightly more subtle way. A default-rule may indirectly express a need on an attribute (and corresponding visit). Furthermore, when a programmer provided an explicit invoke rule for a visit of a child c, then the programmer must give explicit invoke rules in all clauses that require attributes of this visit of c. This is a policy that we impose, because apparently the programmer had a reason to explicitly specify an invocation of the visit, instead of using the implicit specification.

3.5.2 Rule Ordering

To order the rules, we first create production dependency graphs (PDGs) (Section 1.3.4). Note that we do not need nonterminal dependency graphs, because these are fully implied by the interface of a nonterminal. In comparison to conventional PDGs, the PDGs of RulerCore contain also vertices that represent visits, clauses, and invocations of visits. The graph is also slightly less complex because rules do not depend on synthesized attributes of children, but on the visits of the children that produce these attributes.

In the PDG of a production (thus, a semantics block), there exists a begin and end vertex for each visit in the interface. There exists also a vertex for each clause and each rule. Figure 3.12 lists the dependencies between vertices, and gives an impression of the dependency graph of *Menu* in Figure 3.3. The dependency graph is acyclic, thus the rules can be ordered.

In this sketch, the ovals represent rules, and the square boxes represent the begin and end of visits and clauses. The numbers represent the line numbers[3]. The squares immediately following the root are the begin points of the clauses for that visit. Clauses are constrained by begin and end points of visits. Therefore, branches come together again. Some rules are constrained to visits (notably match rules). For other rules, we have more flexibility in their scheduling. For *Menu*, we did not define clauses for the second visits, hence the implicit clause in the graph.

Note that we consider the edges in the direction of their dependency, not in the direction of the value flow. Thus, p is a successor of q if p needs to be defined before q. To make the distinction clearer, we refer to the direct and indirect successors as *dependencies* and direct and indirect predecessors as *users*.

The acyclic graph represents a partial order between rules, visits and clauses. We turn the partial order in a total order in a number of preprocessing steps. Any total order that satisfies the partial order must result in a semantically equivalent result. However, differences between total orders may affect the performance of the resulting algorithm. In the dependency graph, each rule is associated with a unique vertex.

Step 1: order by visit. First, we determine for each rule to which visit to schedule it. Sometimes, rules can be scheduled to more than one visit. The visit a rule is scheduled to influences the amount of data that has to be transported between visits. The inputs for the rule need to be transported to the visit of the rule, and the result of the rule to the visits where these results are used. The set of visits to which a rule r of a production of nonterminal N

[3] See `https://svn.science.uu.nl/repos/project.ruler.systems/ruler-core/examples/JsMenu.rul`.

source node	destination node
begin visit	end of previous visit
end visit	begin visit, clauses, syn attr def. rules, match and pin rules
clause	begin visit
any rule	begin visit or clause
invoke rule	prev invoke or child, inh attr def. rules
any rule w. rhs	begin visit for inh attrs, loc attrs def., invoke of attr

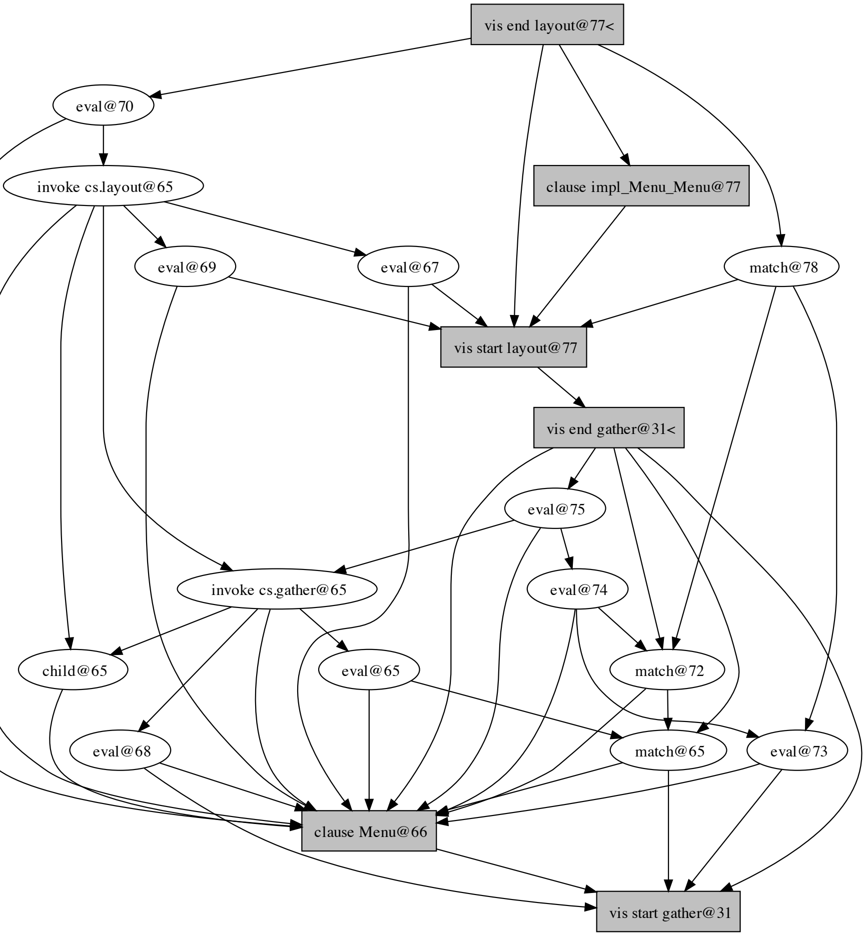


Figure 3.12: Dependencies between RulerCore entities, and the dependency graph of *Menu*.

can be scheduled to are all visits of N, except the visits associated with end-visit nodes in the dependencies of r, and the visits associated with the begin-visit nodes in the users of r.

Given these sets of visits for rules, we can apply scheduling strategies. For example, in Chapter 5, we present notation to specify the iteration of visits. We try to avoid scheduling a rule to such visits, if possible.

We may schedule rule r to any of these visits, however, by doing so, the scheduling may affect the set of possible visits for dependencies and users of r. As our default scheduling strategy, we schedule each rule to its last possible visit[4], and add correspondingly a scheduling dependency from the rule to the beginning of that visit to the graph. This approach has the advantage that the order in which we make decisions for rules does not influence the resulting graph, and also ensures that the graph remains acyclic.

Step 2: order by partition. Secondly, we determine per visit the order of its rules. The order of the rules may affect the amount of overhead in the clause selection. We need to consider the following items.

- Rules are preferably executed once per visit. As heuristic, we schedule rules that do not depend on any clause before the clause selection[5].
- Match-rules and invoke-rules with the partial-strategy may fail. Such rules are preferably scheduled early, because all computations that are performed for a clause that fails is overhead. For example, the match-rules that test the value of an inherited attribute, as introduced by the translation of **datasem**s, should be scheduled upfront.
- Rules may have expressions that are expensive to compute. An accurate estimation of such costs requires an analysis of host-language terms and is hard in general.
- Rules of a particular form may depend on other rules of a different form. We therefore cannot straightforwardly group all match-rules together.

To take these items into account, we use a customizable strategy. We associate with each rule a *partition* and *class* for a finite sequence of partitions and a finite sequence of classes. Rules of one partition are scheduled before rules of a later partition. Within a partition, we schedule rules iteratively. With each iteration, we schedule the rules of the earliest class that has rules that can be scheduled. With this approach, rules of an earlier class precede rules of a later class, if possible. We describe this approach in more detail below.

We define an ordered sequence of partitions. Each rule must be associated with one partition, and a rule may not have a dependency that belongs to a later partition. For a given visit, we schedule the rules of one partition before those of the next partition. The default partitions are:

[4] Each nonterminal has a terminator visit ε, which does not have any attributes, and for which no code is generated. Rules that are scheduled to that visit are discarded during code generation.

[5] An alternative approach is to keep track of which rules that do not depend on clauses are already applied during the evaluation of a previous clause. However, the savings on overhead are likely small. For visits declared as total, such rules eventually need to be applied.

$$\textbf{data}\ P = \mathit{InCommon} \mid \mathit{InClause}\ \ \textbf{deriving}\ (\mathit{Eq}, \mathit{Ord})$$

We associate with *InCommon* all rules of a visit that are not users of a begin-clause-node of that visit, and with *InClause* the remaining rules of that visit.

type of rule	classification	strategy
match-rules	*Earliest*	quick clause selection
partial invoke-rules	*Earlier*	clause selection
pin-rules	*Middle*	early side effect, preferably no backtrack
total invoke-rules	*Later*	visit children when needed
eval-rules	*Latest*	minimize distance between computation and use

Figure 3.13: Association of rules with classes.

Step 3: order by classification. For each partition, we define the order of the rules based on a classification of the rule. Each rule must be associated with a class, which represents a scheduling preference. By default, we distinguish the following classes:

$$\textbf{data}\ C = \mathit{Earliest} \mid \mathit{Earlier} \mid \mathit{Middle} \mid \mathit{Later} \mid \mathit{Latest}\ \ \textbf{deriving}\ (\mathit{Eq}, \mathit{Ord})$$

Figure 3.13 shows the default association based on a rule's type. To influence the ordering, a programmer can specify more classes, and explicitly specify such a class for some of the rules.

The classes specify a scheduling preference: a rule of an earlier class is preferably scheduled before a rule of a higher class. However, this is not possible when a rule of an earlier class depends on a rule of a later class. Moreover, to schedule some rules of an earlier class, there may be different *minimal* subsets of rules of a later class possible, so that it is not obvious which subset to take.

In the following example, the first match-rule is scheduled first. None of the other match-rules can be scheduled, unless either the rule for $loc.a$ or the rule for $loc.b$ is scheduled before. To ensure a deterministic scheduling, we schedule both $loc.a$ and $loc.b$ rules, and use their order of appearance as final distinguishing factor:

```
match loc.x = 3        -- class: Earliest
loc.a = e              -- class: Latest
loc.b = loc.x          -- class: Latest
match loc.y = f loc.a  -- class: Earliest
match loc.z = g loc.b  -- class: Earliest
```

However, if the rule for $loc.y$ would additionally mention $loc.z$ in its RHS, then the rule for $loc.a$ is scheduled directly after the rule for $loc.x$, followed by the rule for $loc.z$, and only then the rule for $loc.b$.

Figure 3.14 shows the scheduling algorithm *rankVertices*. It uses the rank monad R, which is a combination of a list, state, and continuation monad with monad comprehensions [Middelkoop, 2011c], to describe algorithms with the enumeration combinator *foreach* and the

```
  -- gives each vertex of verts a unique rank based on classes
rankVertices :: [C] → [Node] → R ()
rankVertices classes verts = do
  c ← foreach classes
  iter $ do
    vs ← unionM [ps | v ← verts, hasClass v c, let ps = deps′ verts v
                     , allM (λn → hasClass n c ‘impliesM‘ isRanked n) (ps \\ [v])]
    guard (notNull vs)
    iter $ do
      vss ← filterM notNull $ mapM (readyNodes verts vs) classes
      guard (notNull vss)
      v   ← foreach $ sortAsc cmpNodePos $ head vss
      rank v
  -- returns the subset of vs that can be scheduled
readyNodes :: [Node] → [Node] → C → R [Node]
readyNodes verts vs c =
  [v | v ← vs, isNotRanked v, hasClass v c, allM isRanked (deps verts v)]
  -- returns v and its indirect dependencies
deps′ :: [Node] → Node → [Node]
deps′ verts v = [v] ∪ map (deps′ verts) (deps verts v)


iter       :: R () → R ()                  -- iterate param until a guard is False
guard      :: Bool → R ()                  -- check a condition; fail if False
foreach    :: [a] → R a                    -- repeat cont. for each item in list
rank       :: Node → R ()                  -- gives the node the next rank
deps       :: [Node] → Node → [Node]       -- direct dependencies
hasClass   :: Node → C → R Bool            -- True iff the node has the given class
isRanked   :: Node → R Bool                -- True iff the node is ranked
sortAsc    :: (Node → Node → Ordering) → [Node] → R [Node]
cmpNodePos :: Node → Node → Ordering
```

Figure 3.14: Order algorithm for the visit's rules.

iteration combinator *iter*. As parameters, it gets the sequence of classes in the order from earliest to latest, and the set of vertices of the partition to schedule. The result of the algorithm is an association with a unique rank for each of these vertices.

With this algorithm, we try to schedule rules in increasing class order by giving the vertices of such rules an increasing rank. The class c is the class we currently schedule for. For class c, we determine the set *vs*, which is the smallest set that contains all vertices of rules of class c that do not depend on an unranked vertex of class c or an earlier class. Moreover, the set contains the indirect dependencies of these rules. This set may include vertices of a later class, or already ranked vertices. Thus, *vs* contains the largest set of vertices of class c that can be ranked when we only consider already ranked vertices of the same class or earlier. Also, it contains the smallest set of vertices of a later class that need to be ranked to allow the ranking of all the vertices of an earlier class in *vs*. For an acyclic graph, and as long as there are unscheduled rules of class c, the set *vs* has at least one rule. We ensure that each rule in *vs* is scheduled. Thus, by repeating this process we rank all rules of class c. Consequently, after processing all classes, all vertices in *verts* are ranked.

We rank the vertices of *vs* in iterations. With *readyNodes* we determine for each class the set of unranked vertices that have ranked dependencies, and take the nonempty subset of the earliest class. Such a set exists, unless all vertices of *vs* are ranked. We rank these vertices in the order of the appearance of their rules. The algorithm guarantees that an unranked node is ranked if all its dependencies are ranked (soundness property), and that there is per iteration at least one vertex ranked as long as there are unranked vertices left (progress property).

Assume the number of vertices to be ranked is n, and the number of classes is c. Both the worst-case asymptotic time and memory complexity of $deps'$ is $O(n)$, where $|vs| \leqslant n$. The functional implementation assumes that vertices are represented as an integer. The corresponding rule can be determined in constant time via a lookup table. For *readyNodes* the time and memory complexities are $O(n^2)$. The time complexity of the algorithm for *vss* is thus $O(cn^2)$, and of the algorithm for *vs* is $O(n^2)$. The memory complexity is $O(n^2)$. Despite the double iter-blocks, each line is at most repeated $n+c$ times, because with every repetition at least one vertex is ranked. The worst-case time complexity of *rankVertices* is thus $O(cn^3)$ and the worst-case memory complexity is $O(n^2)$. Note that in practice the values for c and n are small, and that most rules have only a small number of dependencies.

Step 4: cleanup. We schedule the rules in a partition in the order of their rank. For the purely functional eval-rules, we make an exception. Starting with the highest-ranked eval-rule, we shift it just before the lowest-ranked of its users in the dependency graph. Scheduling such rules later does not affect the overhead of clause selection, and our strategy is to bring such rules close to where their results are used. Since eval-rules have the *Latest*-classification, it is already likely that these rules are already at such a position. In a similar way, we could also move total invoke-rules. However, the visit invoked by total invoke-rules may contain pin-rules, thus for such rules we keep the scheduling via the classification mechanism.

Remarks. The implicit invokes and automatic ordering allow a straightforward transformation from a datasem-block to a sem-block. Essentially, a datasem-block is syntactic sugar for a number of clauses, each with a match-rule, and a number of child-rules. We also allow in RulerCore omitted visits and clauses to be implicitly defined. Combined with implicit invocations, this makes it easy to add additional visits to an interface. Furthermore, the automatic rule ordering allows us to write independent rules separately from each other (possibly in separate files) and use a preprocessing step to merge the rules together.

The scheduling also offers opportunities to exploit parallelism. When the *head vss* consists of total invoke-rules, during *rankVertices*, these invocations are candidates to evaluate in parallel because their computations are independent. However, whether or not parallelism is beneficial depends on how expensive the computation of the visit is. With the classification mechanism, a rough static approximation can be expressed by the programmer. For example, we can introduce a class for total invocation rules that take priority over conventional invocation rules. When multiple of these invocation-rules are scheduled together, we can actually generate code that performs the visits in parallel.

3.6 Discussion

RulerCore can be used to express traversals over tree-like data structures. To a limited extend RulerCore may be applicable to graphs traversals that are technically tree traversals (such as a traversal over a depth-first forest). Loops and iteration can be expressed with higher-order attributes. In related work, we expressed these by iterating visits [Middelkoop et al., 2010a]). However, RulerCore is not suitable to express traversals over drastically changing data structures.

In our actual implementation, we also provide a notion of internal visits. A conventional visit is invoked externally by the parent, and can choose a clause. This means that we can only conditionally compute attributes once per visit. In contrast, an internal visit is invoked at the end of the clause, and is not visible externally. An internal visit may again have clauses, and these clauses may again specify an internal visit as next visit, or a conventional visit. With this relatively simple extension, we can arbitrarily often branch inside a visit.

In the Haskell version of RulerCore, we require type signatures for attributes. In JavaScript, instead of type signatures, the notion of a type signature represents a dynamic check in the form of assertion-functions that validate the values for attributes.

To fully enjoy the benefits of attribute grammars, the host language requires support for purely functional data structures [Okasaki, 1998]. Such data structures can be encoded in JavaScript, but efficient versions with copy-on-write semantics are cumbersome to implement manually.

3.7 Related Work

Related to this chapter are various visitor-like approaches and attribute grammar techniques.

The purpose of the Visitor design pattern [Gamma et al., 1993] is to decouple traversal operations from the specification of the tree to be traversed, in order to make it easier to add new operations without changing the existing specification of the tree. This allows us to write a multi-visit traversal using a separate visitor per traversal. Multi-methods [Chambers and Leavens, 1994] are supposed to replace the visitor pattern. A multi-method allows overloading of methods on multiple parameters, and makes *accept*-methods superfluous. This, however, is orthogonal to the problems and solutions that we presented in this chapter.

In Section 3.2.1, we discussed advantages and disadvantages of modeling traversals with visitors. In particular, side effects are permitted, and used to store results for use in later visits. The side effects make it hard to predict if results needed in a next visit are actually stored by a first visit. This is a fundamental problem of visitors. Oliveira et al. [2008], for example, show many enhancements with respect to the type safety of visitors, but do not address the transfer of results between visits.

Attribute grammars [Knuth, 1968, 1990] are considered to be a promising implementation for compiler construction. Several attribute grammar techniques are important for our work. Kastens [1980] introduces ordered attribute grammars. In OAGs, the evaluation order of attribute computations as well as attribute lifetime can be determined statically, allowing severe optimizations.

Boyland [1996] introduces conditional attribute grammars. In such a grammar, semantic rules may be guarded. A rule may be evaluated if its guard is satisfied. Evaluation of guards may influence the evaluation order, which makes the evaluation less predictable. In comparison, in our clauses-in-visits model, we have to explicitly indicate in which visits guards are evaluated (the match-statements of a clause), which makes it clear what the evaluation order is. Our approach has the additional benefit that children may be conditionally introduced and visited.

Recently, many Attribute Grammar systems arose for mainstream languages, such as Silver [Wyk et al., 2008] and JastAdd [Ekman and Hedin, 2007] for Java, and UUAG [Löh et al., 1998] for Haskell. In contrast to the work in this chapter, these systems strictly discourage or disallow the use of side effects. The design of RulerBack is inspired by the language of execution plans of UUAG. In certain languages, AGs can be implemented via meta-programming facilities, which obliviates the need of a preprocessor. Viera et al. [2009] show how to implement AGs in Haskell through type level programming. The ideas that we presented in this chapter are orthogonal to such approaches, although the necessary dependency analysis may be difficult to express depending on the expressiveness of the meta language.

3.8 Conclusion

We introduced the language RulerCore, an extension of Attribute Grammars that makes visits to nonterminals explicit. As a consequence, it is possible to use side effects in rules. RulerCore combines the freedom of visitors as described by the visitor design pattern with the convenience of programming with attributes, as shown in Section 3.2.

Moreover, we presented RulerBack, a subset of RulerCore, which serves as a small core language for visitor-based Attribute Grammars. In RulerBack, the lifetime of attributes is explicit, as well as the evaluation order of rules and visits to children. We described how RulerCore programs are mapped to RulerBack in Section 3.5. A RulerBack program has a straightforward translation to many languages. In Section 3.4, we gave a translation to JavaScript by making use of exceptions in combination with the try-catch mechanism. A more sophisticated translation is possible that does not require exceptions, as we show in later chapters.

Future work. A direction of future work is to consider destructive updates on attributed trees. Event-handling traversals over data structures may need to respond to dynamic changes induced by user input or external events. In RulerFront, the visits performed on an attributed tree explicitly specify which attributes are defined. When we apply a destructive update to the tree, we thus know precisely what information is based upon the previous structure of the tree. This knowledge can be exploited to reason about mutations of the attributes tree. Incremental evaluation of attribute grammars [Vogt et al., 1991, Yeh and Kastens, 1988] may be used to efficiently recompute attributes after modifications of the AST.

This chapter treated the scheduling of rules in the presence of side effects. Side effects are not visible in value dependencies between attributes. In Chapter 9 we show how to incorporate dependently-typed programming languages, which have type assumptions resulting from pattern matches as 'effect'.

3.A The Ranking Monad

Figure 3.14 uses the ranking monad R, which we define in this section[6] as background material. These definitions require the `Control.Monad` module. R is a continuation-based monad with failure. We define similar monads in Section 2.4, Section 2.5 and Chapter 5.

Internally, R is a wrapper around a function that threads a state S. We leave S unspecified. This function takes a continuation as parameter, which gets the result of the monadic computation and an updated state. Failure is expressed with a *Maybe* result value:

```
data R a where
  R :: (∀ b.((a → S → (Maybe b, S)) → S → (Maybe b, S))) → R a
instance Functor R where
  fmap f (R h)     = R (λc → h (c.f))
instance Monad R where
  return x         = R (λc   → c x)
  R g >>= f        = R (λc   → g (λa → case f a of R h → h c))
  fail _           = throwError ()
instance MonadState S R where
  get              = R (λc s → c s s)
  put s            = R (λc _ → c () s)
instance MonadError () R where
  throwError ()    = R (λ_ s → (Nothing, s))
  catchError m h   = R (λc s → let f c Nothing s = runR (h ()) s (g c)
                                    f c (Just x) s = c x s
                                    g c Nothing s = (Nothing, s)
```

[6] Code: `https://svn.science.uu.nl/repos/project.ruler.papers/archive/RankMonad.hs`

```
                                 g c (Just x)  s = c x s
                               in runR m s (f c))
runR :: R a → S → (Maybe a → S → b) → b
runR (R f) s g = case f (λx s → (Just x, s)) s of (o, s′) → g o s′
```

The function *foreach* applies the continuation to each element of the list and threads the state through these computations. The result is the outcome of the last succeeding computation:

```
foreach :: [a] → R a
foreach xs = R (λc s → let f c (o, s₀) x = case c x s₀ of
                                            (Nothing, s′) → (o,  s′)
                                            (o′,      s′) → (o′, s′)
                       in foldl (f c) (Nothing, s) xs)
```

The remaining API functions can be defined in terms of the above functions. The *iter* function runs the computation *m* that it takes as parameter until a guard of *m* fails:

```
iter :: R () → R ()
iter m = (m ≫ iter m) ‘orElse‘ ()
orElse :: R a → a → R a
orElse m r = m ‘catchError‘ (λ_ → return r)
guard :: Bool → R ()
guard g = when (¬ g) (throwError ())
```

The function *foreachL* is used in the translation of the monadic list comprehensions, which are not to be confused with monad comprehensions:

```
foreachL :: [R a] → R [a]
foreachL   = foldM f [ ] where
  f acc m = do acc′ ← singleR m ‘orElse‘ [ ]
               return (acc ++ acc′)
singleR :: R a → R [a]
singleR m = m ≫= return.return
```

4 AGs with Commuting Rules

In Chapter 3 we described a programming model based on visits as an extension of attribute grammars. Using this model, attribute declarations are not as easily composed as with conventional attribute grammars. This chapter addresses this issue. We show how to generalize visits to phases. This generalized model is more convenient for programming.

Further, we present commuting rules, which are rules that are connected via a chained attribute[1], but which do not depend on previous rules in the chain. With these rules we can describe the threading of an attribute that follows the implicit visit order of the phases model. To preserve referential transparency, the commuting rules must satisfy a liberal commutativity law.

The extensions of subsequent chapters generalize straightforwardly to phases, although the description is more convenient in terms of visits. Therefore, we describe our extensions in later chapters in terms of visits without loss of generalization.

4.1 Introduction

In this chapter, we show three related extensions of the explicit visit-approach in Chapter 3 and ordered attribute grammars in general.

Firstly, we show how to generalize visits to more declarative *phases*. Computations for a nonterminal symbol take place in one or more distinct phases. We define later that the phase interface of a nonterminal is a set of declarations of these phases. Attribute declarations and rules may be constrained to a phase. The phases are subjected to a partial order. When the value-dependencies between attributes and rules and the dependencies induced by phases are acyclic, the attributes that are associated to a previous phase are defined before any attributes that are associated with a later phase. An evaluation algorithm can be associated with a phase. In this chapter, we assume a statically ordered evaluation algorithm that uses one or more implicit visits per phase.

Secondly, we present a Kennedy and Warren [1976] style scheduling algorithm that infers multiple visit-interfaces from the phase-interface of a nonterminal. For each visit-interface, a production has a potentially different execution plan. An execution plan specifies an explicit ordering of rules per visit (Section 1.3.3). Such a plan specifies also for each child which visit-interface is used for its evaluation.

In the generated code, semantic functions are indexed by the choice of the visit interface for which it provides a semantics, and we show how to encode this in a strongly typed, purely functional language. There may be many possible visit interfaces given the phase interface

[1] A chained attribute is an inherited and a synthesized attribute combination that is threaded through the three by default with copy rules.

of a nonterminal. The order constraints as mentioned above can be used to significantly limit the number of possible visit interfaces.

Finally, we present commutable rules, which allows us to functionally encode side effects. A functional encoding of side effects as mentioned in Chapter 3 can be accomplished with a chained attribute that is threaded through each operation, where an operation is a rule or higher-order child. For example, a substitution attribute captures the side effects that result from substitutions during type inference, and an attribute with the type *RealWorld* captures arbitrary I/O. The attribute's threading determines the order in which the side effects are observable in the values of the attribute.

The order imposed by the threading may be too strict. For *commuting* operations, such as a unique identifier dispenser, the order is largely irrelevant. However, if the associated attribute is not threaded carefully, the order may induce accidental cycles in the dependencies of attributes.

Definition (Commutable rule)**.** A *commutable rule* is a rule that threads a chained attribute.

We present *commutable rules*, which are rules that are connected via a chained attribute, but do not depend on previous rules in the chain. Such rules thus provide more freedom in their ordering. To preserve referential transparency, the composition of commutable rules must satisfy a liberal commutativity law. Commutable rules also allow the functional encoding of a side-effectful rule that is scheduled to different implicit visits.

We thus offer a mechanism to explicitly enforce constraints on the evaluation order of attributes, and another mechanism to loosen the constraints imposed by rules. The combination offers convenient ways to declaratively specify side-effectful operations and their algorithmic evaluation order.

The above subjects have in common that the associated scheduling algorithm takes information into account that is not visible in the attribute dependencies induced by the rules. Phases induce ordering constraints based on dependencies between attributes and the lexical scope of rules. Commutable rules require a barrier between rules that commute over an attribute and rules that do not. Therefore, we introduce barrier attributes and dependency rules, which allow the encoding of such additional dependencies.

Definition (Designator)**.** A *designator* represents an attribute occurrence or a rule.

Definition (Dependency rule)**.** A *dependency rule* $d_1 \prec d_2$ specifies that designator d_1 must be scheduled before designator d_2.

Definition (Barrier attribute)**.** A *barrier attribute* is an attribute that can be used as a *designator* (dependee/depender) in combination with dependency rules. The value of a barrier attribute is implicitly defined.

In this chapter, we first work out the concepts of barrier attributes and dependency rules (Section 4.2), and then show how these concepts can be exploited to deal with phases (Section 4.8) and commutable rules (Section 4.9).

```
grammar Tree                                    -- grammar for nonterm Tree
   | Leaf x :: Int                              -- production Leaf
   | Bin  l, r : Tree                           -- production Bin with two nonterms
attr Tree  syn gath :: Int                      -- attributes for nonterm Tree
           inh mini :: Int
           syn repl :: Tree
sem Tree                                        -- semantics for nonterm Tree
   | Leaf lhs.gath = loc.x
          lhs.repl = Leaf lhs.mini              -- replacement tree
   | Bin  lhs.gath = min l.gath r.gath          -- global minimum gathering
          l.mini   = r.gath                     -- crossover between r and l
          r.mini   = l.gath
          lhs.repl = Bin l.repl r.repl          -- replacement tree
```

Figure 4.1: A variant of the Repmin example.

4.2 Example with Barriers

We use Haskell as host language. Figure 4.1 shows a variant of the classical Repmin [Bird, 1984] example, which requires more than one visit to compute the attributes with a statically ordered evaluation strategy. The attribute *repl* contains a clone of the tree with each leaf-child replaced with the minimum of its sibling's subtree. In order to get the *min* attribute of *l*, the *gath* attribute of *r* is needed and vice versa. With statically ordered evaluation, this can be accomplished by first computing the *gath* attribute in a first visit, then compute *min* and *repl* in a later visit.

Barriers. The evaluations for *l.repl* and *r.repl* are independent. For debugging purposes, we may want to specify that the *l.repl* attribute is computed before the *r.repl* attribute. Since the *syn.repl* attribute depends on the *inh.min* attribute, we get the desired behavior when *l.repl* is a dependency of *r.min*. The following dependency rule expresses this dependency:

sem *Tree* | *Bin*   **order** *l.repl* $\prec$ *r.min*

This rule requires that *inh.min* is a dependency of *syn.repl*. Note that the dependencies are transitive.

The dependencies of dependency rules may not be clear when multiple synthesized attributes are involved, nor remain stable after code changes. For this purpose, we introduce a barrier attribute:

attr *Tree*   **inh** *sync* **barrier**

sem *Tree*

```
  | Leaf   order lhs.sync ≺ lhs.repl
  | Bin    order lhs.sync ≺ l.sync
           order lhs.sync ≺ r.sync
           order l.repl   ≺ r.sync   -- actually subsumes lhs.sync ≺ r.sync
```

Barrier attributes are not defined by a rule. Instead, these attributes are defined implicitly. However, we require that for a synthesized barrier attribute y, that there is at least one production with a dependency rule where *lhs*.y occurs as RHS, and at least one production with a child k where $k.y$ occurs as LHS in a dependency rule. For an inherited barrier attribute y, we require the inverse. There is actually no technical reason for this requirement: if the requirement is not met, it should be taken as a warning that a barrier is not used.

Definition (Updatable attribute)**.** An updatable attribute is an attribute containing a reference to a shared state. As an expression, an occurrence o° represents the value of o in the shared state. As a pattern, an occurrence o° means that the shared state at the location pointed to by o is updated with the matched value. An occurrence $o^\times$ in a pattern means that a new shared location is allocated with the matched value and a reference to it stored in o.

We incorporate side-effects in BarrierAG (Chapter 3) in the form of updatable attributes. In the following example an updatable attribute *unq* is created, read from, and updated:

```
attr Tree   inh unq :: IORef Int    -- inherited unique number dispenser
sem Root | Root
  root.unq^× = 1                    -- creates reference and assigns initial value
sem Tree | Leaf                     -- reads and updates reference
  (loc.myId, alhs.unq°) = (alhs.unq°, alhs.unq° + 1)
```

Normally, attribute occurrences in a pattern are at defining positions. However when $\circ$ is used to specify a store to a shared state, the attribute occurrence is at a usage position. Hence *alhs.unq* refers to the inherited attribute *alhs.unq*, and is a dependency of the rule.

Rules that write to a reference are guaranteed to be applied at most once. The order in which these writes take place depends on the scheduling of attributes. When the attribute *loc.myId* is not used, the write actually does not take place. The only guarantee that is given for such a write is that the reference is created and initialized. For more guarantees, barriers can be used to control and specify the order.

Remarks. Barriers and updatable references should be used sparingly. In particular, rules with updatable references are not functional, which breaks equational reasoning. We present them here as an implementation mechanism for phases and commutable rules. With the later property, we can recover equational reasoning. Note that the code generated from an AG may be functional even if the AG description itself is non-functional.

4.3 Core Representation of AGs with Barriers

In this section, we define the semantics of the dependency rules and barrier attributes using the core language BarrierAG, a desugared subset of higher-order AGs. BarrierAG also permits

$I \ ::= \mathbf{attr}\ N\ \overline{a}$	-- attr decls for nonterminal N
$a \ ::= k\ x\ t$	-- attribute declaration with form k, attr name x and type t
$k \ ::= \mathbf{inh} \mid \mathbf{syn}$	-- attribute forms (often also written as identifier)
$t \ ::= \ :: \tau$	-- attribute type (host language)
$\quad \mid \ \mathbf{barrier}$	-- barrier type
$s \ ::= \mathbf{sem}_{\Gamma}\ P : N\ \overline{r}$	-- semantics of a production P of nonterm N in env Γ
$r \ ::= \mathbf{child}\ x : N = f\ \overline{z^{\triangleright}}$	-- (higher order) child declaration
$\quad \mid \ x : p\ \overline{z^{\triangleleft}} = f\ \overline{z^{\triangleright}}$	-- evaluation rule named x with LHS p and RHS f
$\quad \mid \ \mathbf{order}\ d_1 \prec d_2$	-- order declaration
$d \ ::= \mathbf{child}\ x$	-- designates child x
$\quad \mid \ \mathbf{rule}\ x$	-- designates a rule named x
$\quad \mid \ z$	-- designates an attribute occurrence
$z^{\triangleleft} ::= z^{\bullet}$	-- store in occurrence z
$\quad \mid \ z^{\circ}$	-- write to location referenced by z
$\quad \mid \ z^{\times}$	-- store new handle in z
$z^{\triangleright} ::= z^{\bullet} \mid z^{\circ}$	-- respectively read (closed), and deref (open)
$z \ ::= h.c.x$	-- occurrence attr x of c with form h
$h \ ::= k \mid loc$	-- attribute forms (extended with locals)
$c \ ::= lhs \mid loc \mid x$	-- child designators
x, f, p, P	-- identifiers

Figure 4.2: AG core representation.

attributes that have references to a global state as value, which we need later.

Syntax. Figure 4.2 gives the syntax of BarrierAG. BarrierAG serves as a core language for AGs. Hence, we assume that static checks, desugaring, item grouping, and copy rule insertion have been performed (Section 1.3.6 and Section 1.3.12). This core representation serves two purposes: it allows formal reasoning with AGs, and specifying the construction of dependency graphs and execution plans (also see Section 1.3.2).

The main (meta) nonterminals in the (meta) grammar of Figure 4.2 are I (attr-block) and s (semantics blocks). An attr-block consists of a set of attribute declarations. A semantics block provides the rules for a single production. The environment Γ is often left implicit.

Figure 4.3 shows a desugared version of the earlier example in BarrierAG. The context free grammar is translated to terms in the host language, and is not part of the core language. The symbols of the production are represented as higher-order children and local attributes. Evaluation rules are explicitly named, and consist of a function symbol p and f. The evaluation rule represents the function $\overline{z_1} = p\ (f\ \overline{z_2})$. For example, the rule r_3 represents the function $syn.lhs.gath = id\ (f_3\ syn.l.gath\ syn.r.gath)$. The definition of these functions are bound in the environment Γ, which is an annotation of the sem-block. This representation has the

$$\begin{array}{l}
\mathit{sem\_Tree}\ (\mathit{Leaf}\ x) = \mathit{sem\_Leaf}\ x \\
\mathit{sem\_Tree}\ (\mathit{Bin}\ l\ r) = \mathit{sem\_Bin}\ (\mathit{sem\_Tree}\ l)\ (\mathit{sem\_Tree}\ r) \\
\begin{array}{llll}
\textbf{attr}\ \mathit{Tree} & \textbf{syn} & \mathit{gath} & :: \mathit{Int} \\
 & \textbf{inh} & \mathit{mini} & :: \mathit{Int} \\
 & \textbf{inh} & \mathit{sync} & \textbf{barrier} \\
 & \textbf{syn} & \mathit{repl} & :: \mathit{Tree}
\end{array} \\
\mathit{sem\_Leaf}\ \mathit{field\_x} = \\
\quad \textbf{sem}\ \{f_1 = \mathit{field\_x}, f_2 = \mathit{Leaf}\}\ \mathit{Leaf} : \mathit{Tree} \\
\qquad \begin{array}{lll}
r_0 : & \mathit{id}\ \mathit{loc.loc.x}^{\bullet} & = f_1 \\
r_1 : & \mathit{id}\ \mathit{syn.lhs.gath}^{\bullet} & = \mathit{id}\ \mathit{loc.loc.x}^{\bullet} \\
r_2 : & \mathit{id}\ \mathit{syn.lhs.repl}^{\bullet} & = f_2\ \mathit{inh.lhs.mini}^{\bullet}
\end{array} \\
\qquad \textbf{order}\ \mathit{inh.lhs.sync} \prec \mathit{syn.lhs.repl} \\
\mathit{sem\_Bin}\ \mathit{field\_l}\ \mathit{field\_r} = \\
\quad \textbf{sem}\ \{f_1 = \mathit{field\_l}, f_2 = \mathit{field\_r}, f_3 = \mathit{min}, f_4 = \mathit{Bin}\}\ \mathit{Bin} : \mathit{Tree} \\
\qquad \textbf{child}\ l : \mathit{Tree} = f_1 \\
\qquad \textbf{child}\ r : \mathit{Tree} = f_2 \\
\qquad \begin{array}{lll}
r_3 : & \mathit{id}\ \mathit{syn.lhs.gath}^{\bullet} & = f_3\ \mathit{syn.l.gath}^{\bullet}\ \mathit{syn.r.gath}^{\bullet} \\
r_4 : & \mathit{id}\ \mathit{syn.lhs.repl}^{\bullet} & = f_4\ \mathit{syn.l.repl}^{\bullet}\ \mathit{syn.r.repl}^{\bullet} \\
r_5 : & \mathit{id}\ \mathit{inh.l.mini}^{\bullet} & = \mathit{id}\ \mathit{syn.r.gath}^{\bullet} \\
r_6 : & \mathit{id}\ \mathit{inh.r.mini}^{\bullet} & = \mathit{id}\ \mathit{syn.l.gath}^{\bullet}
\end{array} \\
\qquad \textbf{order}\ \mathit{inh.lhs.sync} \prec \mathit{inh.l.sync} \\
\qquad \textbf{order}\ \mathit{inh.lhs.sync} \prec \mathit{inh.r.sync} \\
\qquad \textbf{order}\ \mathit{syn.l.repl} \quad \prec \mathit{inh.r.sync} \\
\mathit{ast} = \mathit{sem\_Bin}\ (\mathit{sem\_Leaf}\ 1)\ (\mathit{sem\_Leaf}\ 2)
\end{array}$$

Figure 4.3: Desugared example.

advantage that the rules can be duplicated without duplicating the body of the function.

Semantics. Ultimately, we generate host-language code for the core language term. However, to facilitate reasoning with terms in the core language, we first define an operational semantics[2] in Figure 4.4 which denotes a tree walking automaton (Section 1.3.3). The semantics refers to rules in Figure 4.5, which can be interpreted as a dynamic version of the dependency analysis that we define in the next section or as a specification of a tree-walking automaton (Section 1.3.3). Since we deal with higher-order AGs, the operational semantics describes both the construction and decoration of the AST.

A decorated node of the AST is represented as a tuple (N, P, Γ, H), where N is the associ-

[2] This operational semantics is also implemented as part of the `ruler-interpreter`: `https://svn.science.uu.nl/repos/project.ruler.papers/archive/ruler-interpreter-0.1.tar.gz`.

$$\boxed{\Gamma \vdash \Psi_0\,;\, v_0 \longrightarrow v_1\,;\, \Psi_1}$$

$$\frac{H(\mathbf{child}\,x) = v_0 \qquad \Gamma_0 \vdash \Psi_0\,;\, v_0 \longrightarrow v_1\,;\, \Psi_1}{\Gamma_0 \vdash \Psi_0\,;\,(N,P,\Gamma_1,H) \longrightarrow (N,P,\Gamma_1,\{\mathbf{child}\,x \mapsto v_1\} \cup H)\,;\,\Psi_1}\ \text{E.DESCENT}$$

$$\frac{H(\mathit{inh.x.y}) = v \qquad H(\mathbf{child}\,x) = (N',P',\Gamma_2,H') \qquad N'\,;P'\,;H' \surd \mathit{inh.lhs.y}}{\Gamma_0 \vdash \Psi;(N,P,\Gamma_1,H) \longrightarrow (N,P,\Gamma_1,\{\mathbf{child}\,x \mapsto (N',P',\Gamma_2,\{\mathit{inh.lhs.y} \mapsto v\} \uplus H')\} \cup H)\,;\,\Psi}\ \text{E.INH}$$

$$\frac{H(\mathbf{child}\,x) = (N',P',\Gamma_2,H') \qquad H'(\mathit{syn.lhs.y}) = v \qquad N;P;H \surd \mathit{syn.x.y}}{\Gamma_0 \vdash \Psi;(N,P,\Gamma_1,H) \longrightarrow (N,P,\Gamma_1,\{\mathit{syn.x.y} \mapsto v\} \uplus H)\,;\,\Psi}\ \text{E.SYN}$$

$$\frac{(\mathbf{child}\,x : N' = f\ (\overline{z^{\triangleright}})) \in \mathit{rules}\ (N,P) \qquad N;P;H \surd \mathbf{child}\,x \qquad \overline{\Psi;H \vdash z^{\triangleright} \uparrow v}}{\Gamma_0 \vdash \Psi;(N,P,\Gamma_1,H) \longrightarrow (N,P,\Gamma_1,\{\mathbf{child}\,x \mapsto (\Gamma_0\,\Gamma_1\,f)\ \overline{v}\} \uplus H)\,;\,\Psi}\ \text{E.CHILD}$$

$$\frac{\begin{array}{c}(x :\ p\ (\overline{z_1{}^{\triangleleft}}) = f\ (\overline{z_2{}^{\triangleright}})) \in \mathit{rules}\ (N,P) \qquad N;P;H_0 \surd \mathbf{rule}\,x \\ \overline{\Psi_0\,;H_0 \vdash z_2{}^{\triangleright} \uparrow v_0} \qquad \overline{v_1} = (\Gamma_0\,\Gamma_1\,p\ \ .\ \Gamma_0\,\Gamma_1\,f)\ \overline{v_0} \\ \overline{H_0 \vdash v_1 \downarrow z_1{}^{\triangleleft} \rightsquigarrow H;\Psi;\Upsilon} \qquad \Upsilon_1 \uplus \ldots \uplus \Upsilon_n \uplus \mathit{dom}\,\Psi = \overline{\ell}\end{array}}{\Gamma_0 \vdash \Psi_0\,;\,(N,P,\Gamma_1,H_0) \longrightarrow (N,P,\Gamma_1,\overline{H} \uplus \{\mathbf{rule}\,x \mapsto \iota\} \uplus H_0)\,;\,\overline{\Psi}\,\Psi_0}\ \text{E.EVAL}$$

$$\frac{\mathit{syn.y} \in \mathit{barriers}\ N \qquad N;P;H \surd \mathit{syn.lhs.y}}{\Gamma_0 \vdash \Psi;(N,P,\Gamma_1,H) \longrightarrow (N,P,\Gamma_1,\{\mathit{syn.lhs.y} \mapsto \iota\} \uplus H)\,;\,\Psi}\ \text{E.BAR.LHS}$$

$$\frac{(\mathbf{child}\,x : N' = f\ (\overline{z^{\triangleright}})) \in \mathit{rules}\ (N,P) \qquad \mathit{inh.y} \in \mathit{barriers}\ N' \qquad N;P;H \surd \mathit{inh.x.y}}{\Gamma_0 \vdash \Psi;(N,P,\Gamma_1,H) \longrightarrow (N,P,\Gamma_1,\{\mathit{inh.x.y} \mapsto \iota\} \uplus H)\,;\,\Psi}\ \text{E.BAR.INH}$$

Figure 4.4: Small-step evaluation rules for a node.

ated nonterminal, P is the associated production, Γ is an environment that contains bindings for the functions that are mentioned in the rules of P, and the heap H contains values for attributes and nodes that form the children of P.

More precisely, a heap H is a partial map between descriptors d and a value v. A descriptor is either an identifier for an attribute or a child:

$$\begin{array}{lll}
v & ::= \mathsf{v} \mid \iota \mid n \mid \ell & \text{-- atomic value } \mathsf{v}\text{, or unit value } \iota\text{, or node } n\text{, or reference } \ell \\
n & ::= (N,P,\Gamma,H) & \text{-- node associated to } N \text{ and } P\text{, with env } \Gamma\text{, and heap } H \\
H & ::= \emptyset \mid H, d \mapsto_{\delta} v & \text{-- heap of a node (partial map from descriptor to value)} \\
\Psi & ::= \emptyset \mid \Psi, \ell \mapsto\ \ v & \text{-- threaded heap (partial map from reference to value)} \\
\Upsilon & ::= \emptyset \mid \Upsilon, \ell & \text{-- set of references } \overline{\ell}
\end{array}$$

A value of an attribute is either an atomic value v in the domain of the attribute, or a tree in case of a higher-order attribute.

Definition (Normal form)**.** A tree n is in normal form when the synthesized attributes of the root have been computed, thus when the root node n has a heap with bindings for all $syn.lhs.x$ where $syn.x$ is a synthesized attribute declared for the nonterminal of n.

The small-step relation $\Gamma_0 \vdash \Psi\,;\, v \longrightarrow v'\,;\, \Psi'$ (Figure 4.4) describes how to evaluate a tree v one step further to v' in a global environment Γ_0. The threaded heap Ψ contains bindings for references, and is threaded through the evaluation.

To reduce a tree, we start with an initial tree and gradually grow the tree by modifying the heaps. In the initial tree $v_0 = (N, P, \Gamma, \emptyset)$, N is the nonterminal of the root, P is the production of the root, and Γ contains bindings for functions as mentioned in the rules of P. In particular, for each child-rule in P, there is a binding for a function in Γ that provides the semantics for that child. When the AG is well-formed, the resulting tree is in normal form when the rules are applied exhaustively.

We use the following notation in the rules. The operator $\cup$ represents the left-biassed union of its two operands heaps, and operator $\uplus$ takes the union of two heaps with disjoined keys. The domain $dom\ (H)$ of a heap H is the set of the heap's keys. A heap H applied to a designator d, written $H\ (d)$, returns the binding for d in H. Similarly, $\Gamma_0\ \Gamma_1\ f$ returns the function binding for f in the right-biassed union of Γ_0 and Γ_1. Juxtaposition of heaps represents the left-biassed union.

In Figure 4.4, rule E.DECENT incorporates the idea of nondeterministically selecting a node constructed so far, and reducing it a bit further. Each node has its own heap that explicitly tracks which attributes have been computed, which children have been constructed, and which rules have been applied. A test to check if a rule can be applied then boils down to verifying that their dependencies are met (Figure 4.5).

With Rule E.INH bindings for the inherited attribute of a child can be copied to the heap of that child when the child is ready to receive these bindings. For example, inherited attributes for a child can be computed even before the child is introduced. Rule E.SYN represents the inverse for synthesized attributes. Note the careful use of disjoined unions which ensure that a rule is only applicable once per attribute.

With rules E.CHILD, E.EVAL, E.BAR.INH, and E.BAR.SYN, bindings can be added to the heap. Evaluation of the RHS of the AG's child-rule gives us the initial tree to use for that child in the heap. Evaluation of the AG's evaluation rule results in bindings for one or more attributes. The rule E.EVAL ensures in addition that potentially newly introduced references Υ do not clash with existing references. The rule E.BAR.INH defines an inherited barrier attribute of a child x. Rule E.BAR.SYN defines a synthesized barrier attribute of lhs.

Figure 4.5 shows load, store, and ready rules. The relation $\Psi; H \vdash z^{\triangleright} \uparrow v$ represents a load of value v using designator z in heap H. The load rule L.VAL requires the presence of a designator in the heap. With the rule the value of the designator can be extracted from the head. The load rule L.REF represents an indirect load from the threaded heap Ψ.

The relation $H_0 \vdash v \downarrow z \rightsquigarrow H\,;\,\Psi\,;\,\Upsilon$ represents a store of value v. The heap H and treaded heap Ψ contain the new bindings that result from storing v. The initial heap H_0 is stores the reference when $z^{\triangleleft}$ is an indirection. The set of references Υ contains the newly added references. The store rule S.NEW represents the creation and assignment of a new reference ℓ, and S.UPDATE an update via such a reference. With rule S.SAVE a binding for z is added without an indirection.

$$\boxed{\Psi ; H \vdash z^{\triangleright} \uparrow v}$$

$$\frac{H(z) = v}{\Psi ; H \vdash z^{\bullet} \uparrow v}\ \textsc{l.val} \qquad\qquad \frac{H(z) = \ell \qquad \Psi(\ell) = v}{\Psi ; H \vdash z^{\circ} \uparrow v}\ \textsc{l.ref}$$

$$\boxed{H_0 \vdash v \downarrow z^{\triangleleft} \rightsquigarrow H ; \Psi ; \Upsilon}$$

$$H_0 \vdash v \downarrow z^{\times} \rightsquigarrow \{z \mapsto \ell\} ; \{\ell \mapsto v\} ; \{\ell\}\ \textsc{s.new} \qquad\qquad H_0 \vdash v \downarrow z^{\bullet} \rightsquigarrow \{z \mapsto v\} ; \emptyset ; \emptyset\ \textsc{s.save}$$

$$\frac{H(z) = \ell}{H_0 \vdash v \downarrow z^{\circ} \rightsquigarrow \emptyset ; \{\ell \mapsto v\} ; \emptyset}\ \textsc{s.update}$$

$$\boxed{N ; P ; H \surd d}$$

$$\frac{N ; P ; H \surd_{\text{dep}}\, d \qquad \{d' \mid (\textbf{order}\ d' \prec d) \in \mathit{rules}\ (N,P)\} \subseteq \mathit{dom}\ H}{N ; P ; H \surd d}\ \textsc{d.order}$$

$$\frac{\begin{array}{c}(x\colon p\ \overline{z_1{}^{\triangleleft}} = f\ \overline{z_2{}^{\triangleright}}) \in \mathit{rules}\ (N,P) \\ \overline{N ; P ; H \surd z_1}\end{array}}{N ; P ; H \surd_{\text{dep}}\, \textbf{rule}\ x}\ \textsc{d.r} \qquad \frac{}{N ; P ; H \surd_{\text{dep}}\, \textbf{child}\ x}\ \textsc{d.c} \qquad \frac{}{N ; P ; H \surd_{\text{dep}}\, z}\ \textsc{d.a}$$

Figure 4.5: Load, store and designator dependency rules.

The relation $N ; P ; H \surd d$ describes when d is ready to be defined while taking dependency-rules into account. These rules do not test for the existence of values in the heap, because the load and store rules already cover for this. The rule D.ORDER handles dependency-rules generically for the designators. The rule D.R states that an AG rule is ready when its LHS is ready.

Remarks. If we can determine for a nonterminal that no subtree applies rule S.NEW or S.UPDATE, only topdown behavior for Ψ is needed. If in addition L.REF cannot occur, the two Ψ parameters can be omitted from the relation.

The rule E.DESCENT facilitates a walk down the tree until the node to reduce is reached. An actual implementation performs multiple evaluation steps (if possible) on such a node in order to reduce traversal overhead. In fact, it is possible to statically analyze the AG and obtain a static scheduling of the rules.

$$
\begin{aligned}
&\llbracket \textbf{sem}\, P : N\, \overline{r} \rrbracket_{\mathsf{vert}} &&= \{\textbf{child}\, \mathit{lhs}\} \cup \llbracket \overline{a}_N \rrbracket_{\mathsf{vert}\,(\mathit{lhs})} \cup \llbracket \overline{r} \rrbracket_{\mathsf{vert}} \\
&\llbracket \textbf{inh}\, y\, t \rrbracket_{\mathsf{vert}\,(x)} &&= \{\mathit{inh}.x.y\} \\
&\llbracket \textbf{syn}\, y\, t \rrbracket_{\mathsf{vert}\,(x)} &&= \{\mathit{syn}.x.y\} \\
&\llbracket \textbf{child}\, x : N = f\, \overline{z^{\triangleright}} \rrbracket_{\mathsf{vert}} &&= \{\textbf{child}\, x\} \cup \llbracket \overline{a}_N \rrbracket_{\mathsf{vert}\,(x)} \\
&\llbracket x :\, p\, \overline{z_1{}^{\triangleleft}} = f\, \overline{z_2{}^{\triangleright}} \rrbracket_{\mathsf{vert}} &&= \{\textbf{rule}\, x\} \\
&\llbracket \textbf{order}\, d_1 \prec d_2 \rrbracket_{\mathsf{vert}} &&= \emptyset
\end{aligned}
$$

$$
\begin{aligned}
&\llbracket \textbf{sem}\, P : N\, \overline{r} \rrbracket_{\mathsf{edge}} &&= \llbracket \overline{r} \rrbracket_{\mathsf{edge}} \\
&\llbracket \textbf{child}\, x : N = f\, \overline{z^{\triangleright}} \rrbracket_{\mathsf{edge}} &&= \overline{z \leftarrow \textbf{child}\, x} \cup \{\textbf{child}\, x \leftarrow \textbf{syn}.x.y \mid \textbf{syn}.y \in a_N\} \\
&\llbracket x :\, p\, \overline{z_1{}^{\triangleleft}} = f\, \overline{z_2{}^{\triangleright}} \rrbracket_{\mathsf{edge}} &&= \overline{z_2 \leftarrow \textbf{rule}\, x} \cup \llbracket z_1{}^{\triangleleft} \rrbracket_{\mathsf{pre}\,(x)} \\
&\llbracket \textbf{order}\, d_1 \prec d_2 \rrbracket_{.} &&= \{d_1 \leftarrow d_2\} \\
&\llbracket z^{\bullet} \rrbracket_{\mathsf{pre}\,(x)} &&= \{\textbf{rule}\, x \leftarrow z\} \\
&\llbracket z^{\times} \rrbracket_{\mathsf{pre}\,(x)} &&= \{\textbf{rule}\, x \leftarrow z\} \\
&\llbracket z^{\circ} \rrbracket_{\mathsf{pre}\,(x)} &&= \{z \leftarrow \textbf{rule}\, x\}
\end{aligned}
$$

Figure 4.6: Vertices and edges of the initial production dependency graph.

4.4 Static Dependency Graphs

In order to obtain a static scheduling of the rules, we first construct dependency graphs in the style of Knuth-1 [Knuth, 1968]. If these graphs are cycle free, a static scheduling of the rules is possible.

Unlike in Section 1.3.2, we mean an augmented PDG when we talk about a PDG. When we talk about the initial PDG, we mean the non-augmented PDG as in Section 1.3.2. Thus, we will be dealing with dependency graphs per production (augmented production dependency graph *pdg* (N,P)) and dependency graphs per nonterminal (a nonterminal dependency graph *ndg* (N)).

We first define the initial graphs, then specify the actual graphs as the fixpoints of the functions:

$$
\begin{aligned}
&\mathit{pdg}\, N\, P = \mathit{pdg}_{\mathsf{init}}\, N\, P \cup \{\mathit{instantiate}\ x\ (\mathit{ndg}\, N_x) \mid x \in \mathit{children}\, P\} \\
&\mathit{ndg}\, N \quad\ = \mathit{ndg}_{\mathsf{init}}\, N \quad \cup \{\mathit{abstract}\ (\mathit{pdg}\, N\, P) \quad\ \mid P \in \mathit{prods}\, N\}
\end{aligned}
$$

The function *instantiate* translates the edges between declarations of N_c in the NDG as edges between the attributes of child c in the PDG. Similarly, when there is a path between two attributes of *lhs* in the PDG, then *abstract* translates these as edges between the same attributes in N's NDG.

Initial production dependency graph. Per production P of nonterminal N, we construct a production dependency graph $\mathit{pdg}_{\mathsf{init}}\ N\ P$. Thus, each sem-block is translated to a PDG.

Figure 4.6 shows that the vertices of $pdg_{\mathsf{init}}\ N\ P$ consists of designators $[\![\mathbf{sem}\ P : N\ \overline{r}]\!]_{\mathsf{vert}}$. In this notation, $\overline{a}_N$ is the set of attribute declarations a of nonterminal N. The directed edges $[\![\mathbf{sem}\ P : N\ \overline{r}]\!]_{\mathsf{edge}}$ of $pdg\ (N, P)$ relate designators. An edge from d_2 to d_1 denotes that d_1 must be defined before d_2. Order rules are thus a means to arbitrarily add edges to the PDG.

The initial PDG does not contain the dependencies imposed by the semantics of each child (e.g. the actual tree). We obtain the actual production dependency graph by augmenting it with the edges taken from the nonterminal dependency graphs of the children which are an approximation of the dependencies of such trees.

Initial nonterminal dependency graph. The vertices of the nondeterminal dependency graph $ndg\ (N)$ are the declarations a of N:

$$\begin{aligned}
[\![\mathbf{attr}\ N\ \overline{a}]\!]_{\mathsf{vert}} &= [\![\overline{a}]\!]_{\mathsf{vert}} \\
[\![\mathbf{inh}\ x\ t]\!]_{\mathsf{vert}} &= \{ inh.x \} \\
[\![\mathbf{syn}\ x\ t]\!]_{\mathsf{vert}} &= \{ syn.x \}
\end{aligned}$$

The initial NDG does not have any edges. These edges are inferred from the PDGs, as we see below.

Graph construction. The actual PDG is the least solution to the above equations with the following definitions for *abstract* and *instantiate*. The relation $d_1 \leftarrow^{+} d_2$ represents a path from d_2 to d_1, with $d_1 \not\equiv d_2$:

$$\begin{aligned}
abstract\ g &= \{ k_1.x \leftarrow k_2.y \mid k_1.lhs.x \leftarrow^{+} k_2.lhs.y \in g \} \\
instantiate\ x\ g &= \{ k_1.x.y \leftarrow k_2.x.z \mid k_1.y \leftarrow^{+} k_2.z \in g \}
\end{aligned}$$

Conventionally, a synthesized attribute of a child only depends on an inherited attribute of a child. However, due to dependency-rules, any attribute of a child can potentially depend on any other attribute of the child, hence our definition of *abstract* and *instantiate* take these also into account.

The construction of the graphs is a relatively straightforward fixpoint computation. As intermediate data structure, a transitively closed PDG allows for efficient tests for paths between vertices.

Example. In the Leaf-production, there is dependency of *syn.lhs.repl* on *inh.lhs.mini* via rule r_2. This thus induces a dependency of *syn.repl* on *inh.mini* in the NDG. Figure 4.7 shows the PDG of the production *Bin*. The solid edges are initial edges, and the dashed edges are instantiated from the NDG. The Bin-production by itself does not induce any edges in the NDG. We do not show the PDG of the Leaf-production here; it is shown in Figure 4.15(a).

Properties. A NDG can only be cyclic if at least one of the PDGs is cyclic. The reason is that the edges of the NDG are projected into the PDGs, thus a PDG must have a subgraph with these cyclic edges.

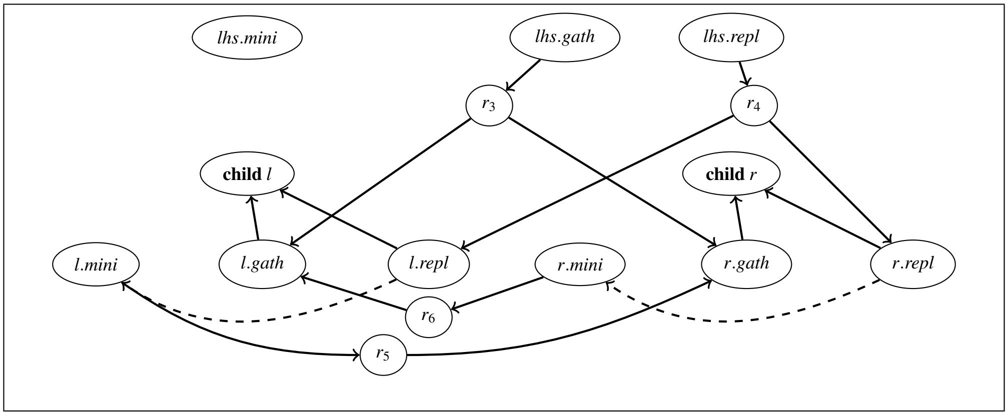


Figure 4.7: Example of the (augmented) PDG of the Bin-production.

When the PDGs are acyclic, then an evaluation to normal form is possible for any well-formed tree. In other words, the graph construction is sound. This property has a relatively straightforward structural induction proof on the shape of the tree. The NDGs symbolize the induction hypothesis. When an evaluation algorithm respects the dependencies of the PDGs, and the constraints of each rule are preserved, the evaluation algorithm is sound. Moreover, the evaluation algorithm is complete if it finds a scheduling when the PDGs are acyclic. We show such an algorithm in Section 4.5.

Rules of a production may make an inherited attribute of a child dependent on a synthesized attribute of a child. These dependencies are not part of the NDG. When we add these edges also to the NDG, each occurrence of a nonterminal has the same dependencies of its inherited attributes on its synthesized attributes. When the PDGs are cycle free without these edges, they are typically also cycle free with these additional edges. In the next section, it becomes clear that adding these edges may reduce the number of visit interfaces that a nonterminal symbol needs to support, thus can be beneficial when the code size is an issue.

4.5 Visits Graphs

An interface for a nonterminal is a partitioning of the attributes into a finite sequence of visits. Given such a sequence, we can produce an execution plan and generate code, as we showed in Chapter 3. We derive such interfaces from acyclic PDGs.

Given such a sequence, it induces a dependency of all attributes of a later visit on attributes of an earlier visit. When represented as scheduling-induced edges in the PDG, it may make the PDG cyclic. It is typically possible to define a single visit sequence per nonterminal that does not make the PDG cyclic. However, it is generally hard to automatically find such a sequence [Kastens, 1980] when the sequence is not manually specified as we presented in Chapter 3. Instead, if we allow multiple interfaces for a nonterminal, we can define visit sequences so that they do not lead to additional dependencies. This is the approach taken by Kennedy and Warren [1976].

$$
\begin{aligned}
[\![\textbf{visit}\ i\ \overline{k.y}]\!]_{\mathsf{vert}\ (x)} &= \{\,\textbf{visit}\ x\ i\,\} \\
[\![\textbf{visit}\ i\ \overline{k.y}]\!]_{\mathsf{edge}\ (x)} &= [\![i]\!]_{\mathsf{edge}\ (x, pre\ i)} \cup [\![\overline{k.y}]\!]_{\mathsf{edge}\ (x,i)} \\
[\![i]\!]_{\mathsf{edge}\ (x,\varepsilon)} &= \{\,\textbf{child}\ x\,\} \\
[\![i]\!]_{\mathsf{edge}\ (x,j)} &= \{\,\textbf{visit}\ x\ j \leftarrow \textbf{visit}\ x\ i\,\} \\
[\![inh.y]\!]_{\mathsf{edge}\ (x,i)} &= \{\,inh.x.y\ \leftarrow \textbf{visit}\ x\ i\,\} \\
[\![syn.y]\!]_{\mathsf{edge}\ (x,i)} &= \{\,\textbf{visit}\ x\ i \leftarrow syn.x.y\,\}
\end{aligned}
$$

Figure 4.8: The embedding of a context.

In our experience, Kastens' approach requires severe manual intervention for large AGs, thus we take the approach by Kennedy and Warren [1976]. However, it is not immediately obvious how to deal with multiple interfaces in a strongly typed language, because the type of the semantic function depends on the interface. In the next section, we solve this problem using type indices in combination with GADTs. In this section, we show how to determine these interfaces.

Contexts. A *context* $\mathcal{C}$ of a nonterminal N, or nonterminal symbol with N, or a production of N is a visit sequence (many v) that is consistent with *ndg* N:

```
C ::= v̄            -- a context consists of a sequence of visits v̄
v ::= visit i k.x   -- a visit consists of a set of attributes k.v and globally unique i
i                   -- identifier
```

A context contains a subset of the attributes declared for a nonterminal. A nonterminal may have a context with less attributes when not all attributes of a nonterminal symbol with this nonterminal are needed.

A visit sequence is consistent with *ndg* N when the following conditions are met.

- For each attribute $inh.x$ and $syn.y$ in the sequence, with $inh.x \leftarrow^{+} syn.y$ in *ndg* N, either $syn.y$ is not part of the visit sequence ,or $syn.y$ is in the same or later visit as $inh.x$.
- For each attribute $syn.x$ and $inh.y$ in the sequence, with $syn.x \leftarrow^{+} inh.y$ in *ndg* N, either $inh.y$ is not part of the visit sequence, or $inh.y$ is in a later visit as $syn.x$.

The number of consistent visit sequences is finite, but potentially large, as it is in the worst case exponential in the number of attributes.

Embedding of a context. We make the context of a child visible in the PDG. For that purpose, we introduce a vertex form that represents a visit:

```
d ::= ...
   | visit x i   -- a visit to x identified by i
```

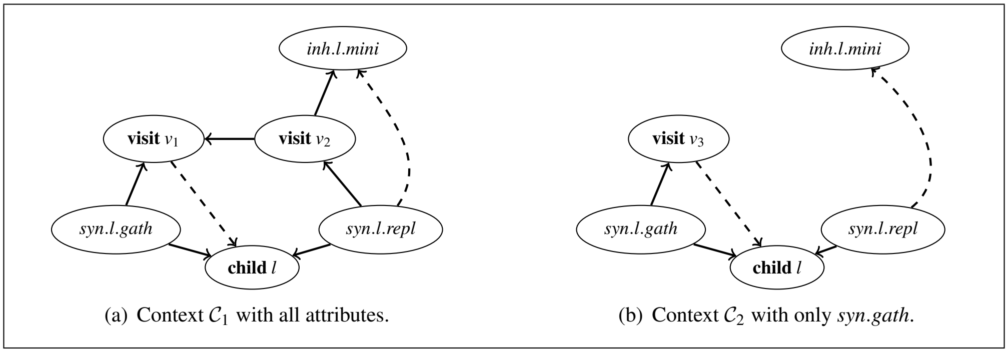


Figure 4.9: Examples of visit-additions to the PDG.

For a child x in context C_x, we add additional vertices and edges to the PDG, which we call the *embedding* of C_x of child x in the PDG. Informally, to embed a context, we add a visit-node and add edges between this node and the associated inherited and synthesized attributes. Additionally, a visit node is dependent on the node of the preceding visit, if any, and the child-node otherwise. Formally, let *pre i* either be the preceding visit j of i, or ε when i is the first visit. We add vertices $[\![C]\!]_{\mathsf{vert}\ (x)}$ and edges $[\![C]\!]_{\mathsf{edge}\ (x)}$ as described in Figure 4.8.

Definition (Visit PDG)**.** For some given contexts of children, a *visit PDG* (VPDG) is the embedding of these contexts in the PDG.

The following are exemplary contexts of child l of production *Bin*. The context $\mathcal{C}_1$ describes the computation of all attributes in two phases. The context $\mathcal{C}_2$ only the computation of *syn.gath*:

$$\begin{aligned}
\mathcal{C}_1 &= [\textbf{visit}\ v_1\ \{syn.gath\}, \textbf{visit}\ v_2\ \{inh.mini, syn.repl\}]\\
\mathcal{C}_2 &= [\textbf{visit}\ v_3\ \{syn.gath\}]\\
\mathcal{C}_3 &= [\textbf{visit}\ v_4\ \{syn.gath, inh.mini, syn.repl\}]
\end{aligned}$$

Figure 4.9 shows the relevant subgraphs after embedding the respective contexts $\mathcal{C}_1$ and $\mathcal{C}_2$. The dashed edges are already existing edges. The embedding of $\mathcal{C}_3$ would lead to a cycle in the PDG.

A visit-node in a VPDG serves as bookkeeping that a visit to the child is needed to compute the attributes it is associated with. It represents dependencies induced from the AG-wide scheduling of attribute computations, and we will ensure later that each visit node has an associated algorithm that computes the attributes.

Abstraction of visit-nodes. We see later as well that the bookkeeping with visits is too verbose, so we establish a more compact representation. For that purpose, we change the syntax of child-vertices. These additionally define in what state $\overline{a}$ the child is, which can be used in the graph to *abstract* from a sequence of visits:

$$\begin{aligned}
d &::= \ldots\\
&\mid\ \textbf{child}\ x\ \overline{a} \quad \text{-- a child } x \text{ in state } \overline{a} \text{ (initially } \emptyset)
\end{aligned}$$

```
$abstract\ g$
    $\mid \langle \textbf{invoke}\ x\ i \rangle \in g \wedge \langle \textbf{child}\ x\ \overline{a} \leftarrow \textbf{invoke}\ x\ i \rangle \in g =$
      $\textbf{let}\ v\ = \langle \textbf{child}\ x\ (\overline{a} \uplus \overline{a}_i)\ \rangle$
          $es = \{ v \leftarrow syn.x.y \mid syn.x.y \in g \} \cup$
                $\{ v \leftarrow \textbf{invoke}\ x\ j \mid \langle \textbf{invoke}\ x\ j \rangle \in g, \langle \textbf{invoke}\ x\ i \leftarrow \textbf{invoke}\ x\ j \rangle \in g \}$
      $\textbf{in}\ abstract\ ((g \cup \{ v \} \cup es) - \{ \textbf{invoke}\ x\ i, \textbf{child}\ x\ \overline{a} \})$
    $\mid otherwise = g$
```

Figure 4.10: Abstraction on a VPDG.

Let g be an acyclic VPDG. Figure 4.10 shows how to abstract from the visit-vertices. We first remove all visit-vertices. When a visit-vertex i is removed, its associated attributes $\overline{a}_i$ are added to the child-vertex. There are no visit-vertices left in g after *abstract* has been applied.

With abstraction applied to a VPDG we obtain scheduled PDGs:

Definition (Scheduled PDG)**.** Given a production P in context $\mathcal{C}$, and context $\overline{\mathcal{C}}$ for its children, its VPDG g is a *scheduled PDG* (SPDG) when:

- The VPDG g is acyclic.
- All reachable synthesized attributes of a child depend on a child node that includes the synthesized attribute in its state. Thus, for each $syn.y \in \mathcal{C}$, for each path in the VPDG from vertex $syn.lhs.y$ to a vertex $syn.x.z$, there exists a vertex **child** $x\ \overline{a}$ with $syn.z \in \overline{a}$.

These two conditions allow us to gradually determine contexts, as we show in the remainder of this section. We show in the next section how to convert a SPDG to an execution plan for a production.

Representation. We determine a set of contexts for each nonterminal, such that there exists a SPDG for each context and each production. Preferably, the set of contexts is small, because each context requires a different semantic function. Also, to save traversal overhead, preferably each context contains visits with many attributes. To determine this set, we use a more sophisticated representation, the visits graph, which we define below.

Note again that a context is a sequence of visits that is associated with a nonterminal N, where each visit describes a transition of the configuration of a node associated with N. Given a set of contexts, sequences of visits may have common intermediate configurations. Thus, we can represent a set of contexts as a graph, which we call the visits graph:

Definition (Visits graph)**.** Given a set of contexts $\overline{\mathcal{C}}$, a *visits-graph* is DAG where the vertices are the union of $[\![\mathcal{C}]\!]_{\mathsf{vert}}$ of each $\mathcal{C} \in \overline{\mathcal{C}}$, and the edges are the union of $[\![\mathcal{C}]\!]_{\mathsf{edge}}$ of each $\mathcal{C} \in \overline{\mathcal{C}}$, as described by Figure 4.11. The vertices represent configurations, and the edges represent visits.

$$
\begin{aligned}
[\![\mathcal{C}]\!]_{\mathsf{vert}} &= [\![\mathcal{C}]\!]_{\mathsf{vert}\,(\emptyset)} \\
[\![\emptyset]\!]_{\mathsf{vert}\,(s)} &= \{s\} \\
[\![\textbf{visit}\ i\ \overline{a} : vs]\!]_{\mathsf{vert}\,(s)} &= \{s\} \cup [\![vs]\!]_{\mathsf{vert}\,(s \cup \overline{a})} \\
[\![\mathcal{C}]\!]_{\mathsf{edge}} &= [\![\mathcal{C}]\!]_{\mathsf{edge}\,(\emptyset)} \\
[\![\emptyset]\!]_{\mathsf{edge}\,(s)} &= \emptyset \\
[\![\textbf{visit}\ i\ \overline{a} : vs]\!]_{\mathsf{edge}\,(s)} &= \{s \xrightarrow{i} (s \cup \overline{a})\} \cup [\![vs]\!]_{\mathsf{edge}\,(s \cup \overline{a})}
\end{aligned}
$$

Figure 4.11: Vertices and edges of the visits-graph.

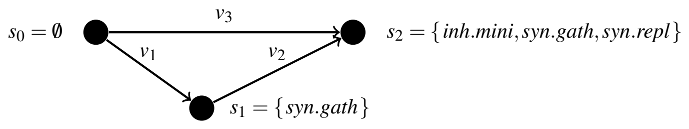


Figure 4.12: Example of a visits-graph which represents contexts $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$.

Figure 4.12 shows the visits-graph that represents the contexts $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$. Note that the context $\mathcal{C}_2$ is fully implied by the other contexts.

For a generated algorithm, a path in the visits-graph represents how a node is visited by its parent. However, a parent may stop invoking visits after any visit. From a code generation perspective, it may be beneficial to know at which vertex a parent stops. We can model this situation in the graph, but we refrain from this complication here.

When two paths in the visits graph converge, this means that two different visit sequences ended up in the same configuration. Consequently, the children are in the same configuration, but were potentially subjected to different visit sequences. The VPDGs may thus differ in the visit-vertices, therefore we need *abstract* (as defined earlier) to eliminate this difference.

We keep a more complex administration for the visits graph, with vertices of the form s and edges of the form v, as described by Figure 4.13. We explain some aspects of this notation below. Figure 4.14 shows this administration for the example.

A vertex of the visits graph represents a context. Per production, a vertex stores the VPDG in this context, and the states of the children. An edge represents a visit, which is a transition from one configuration to one of its next configurations. It stores the attributes $\overline{a}$, which are the inherited attributes provided by the parent for the visit, and the synthesized attributes that need to be computed for the parent. Per production, an edge stores an ordered sequence of invocations to children that are required for the visit. These visits are grouped in what we call *simultaneous visits*. Visits of such a group visit children independently. Note that these visits may be invoked in any order, and may also be invoked concurrently.

Figure 4.15(a) displays the graphs g_1, g_2, g_3 and g_4 that are mentioned in Figure 4.14. The operation *abstract* is not yet applied in the graphs. The dotted edges represent the edges that are inserted due to the embedding of visits. An important property of such a graph is that the

s	$::= \textbf{conf}\, N\, \overline{a}\, \overline{p}$	-- configuration of a node with nonterminal N (key $\overline{a}$)
p	$::= \textbf{prod}\, P\, \overline{k}\, g$	-- production in a context of nonterminal N
k	$::= \textbf{child}\, x : N\, \overline{a}$	-- configuration of child x (key $\overline{a}$)
e	$::= s\, ;\, v\, ;\, s'$	-- edge between s and s'
v	$::= \textbf{visit}\, i : N\, \overline{a}\, \overline{c}$	-- visit i with production info $\overline{c}$ (key i, non-empty $\overline{a}$)
c	$::= \textbf{prod}\, P\, \overline{r}$	-- sequence of invocation sets $\overline{r}$ (possibly empty)
r	$::= \textbf{sim}\, \overline{m}$	-- simultaneous invocations $\overline{m}$ (non-empty, unordered)
m	$::= \textbf{invoke}\, x\, i$	-- invocation of visit i to x (key i)
g		-- VPDG of a production in a given context
N, P		-- identifier (nonterminal), identifier (production)

Figure 4.13: Notation for visits-graphs.

simultaneous visits contain precisely those synthesized attributes that can be visited because the inherited attributes are available (described below with the relation *avail*), and also need to be visited in order to produce values for the synthesized attributes.

The example is rather symmetric. The children of the production *Bin* are treated in the same way. This is in general not the case and will be handled correctly. For example, children may be of different nonterminals, or may be visited in a different order. This situation arises when we use $\langle \textbf{invoke}\, l\, 3 \rangle$ for the left child of the bin-production, or in the example of Section 2.2.

Invariants. The representation of visits graph leads to a recursive set of constraints (presented below) for which we can incrementally construct *a* visits graph. For a given attribute grammar, there may be many possible visits graphs. The way we represent the graph gives rise to visit sequences that are relatively independent and only compute what is necessary, but which are not guaranteed to be as large as possible. Different visits graphs may exhibit small differences in performance, although in principle any visits graph suffices. We impose a number of constraints on the visits graph for which it is possible to find a solution when the PDGs are acyclic, and for which it is possible to fine a *unique* solution.

In the formalization below, we denote a relation as partial function without a right-hand side, or as (partial) boolean functions. Moreover, when used in a boolean expression, we consider such relations a total function from the arguments to a Boolean result value, which is *True* if and only if the arguments form an element of the relation. When the expression of a guard has the value $\perp$, then the guard itself has the value *False*. Also, for unbound variables in pattern expressions we assume existential quantification, unless indicated explicitly otherwise.

Definition (Available)**.** A vertex d is *available* in a VPDG g when vertex d can be scheduled, and respects the order imposed by g. Formally, the relation $avail\ \overline{a}\ g\ d$ as defined in Figure 4.16 (explained below) states whether a vertex d in a VPDG g is available when the node is in configuration $\overline{a}$.

$$
\begin{aligned}
s_0 &= \mathbf{conf}\ \mathit{Tree}\ \emptyset\ \{p_1, p_2\} \\
s_1 &= \mathbf{conf}\ \mathit{Tree}\ \{\mathit{syn.gath}\}\ \{p_3, p_4\} \\
s_2 &= \mathbf{conf}\ \mathit{Tree}\ \{\mathit{inh.mini}, \mathit{syn.gath}, \mathit{syn.repl}\}\ \{p_5, p_6\}
\end{aligned}
$$

$$
\begin{aligned}
p_1 &= \mathbf{prod}\ \mathit{Leaf}\ \emptyset\ g_1 \\
p_3 &= \mathbf{prod}\ \mathit{Leaf}\ \emptyset\ g_1 \\
p_5 &= \mathbf{prod}\ \mathit{Leaf}\ \emptyset\ g_1 \\
p_2 &= \mathbf{prod}\ \mathit{Bin}\ \ \{\mathbf{child}\ l : \mathit{Tree}\ \emptyset, \mathbf{child}\ r : \mathit{Tree}\ \emptyset\} && (\mathit{abstract}\ g_2) \\
p_4 &= \mathbf{prod}\ \mathit{Bin}\ \ \{\mathbf{child}\ l : \mathit{Tree}\ \{\mathit{syn.gath}\}, \mathbf{child}\ r : \mathit{Tree}\ \{\mathit{syn.gath}\}\} && (\mathit{abstract}\ g_3) \\
p_6 &= \mathbf{prod}\ \mathit{Bin}\ \ \{\mathbf{child}\ l : \mathit{Tree}\ \{\mathit{inh.mini}, \mathit{syn.gath}, \mathit{syn.repl}\} \\
&\qquad\qquad\quad , \mathbf{child}\ r : \mathit{Tree}\ \{\mathit{inh.mini}, \mathit{syn.gath}, \mathit{syn.repl}\}\} && (\mathit{abstract}\ g_4)
\end{aligned}
$$

$$
\begin{aligned}
v_1 &= \mathbf{visit}\ 1 : \mathit{Tree}\ \{\mathit{syn.gath}\} && \{c_1, c_2\} \\
v_2 &= \mathbf{visit}\ 2 : \mathit{Tree}\ \{\mathit{inh.mini}, \mathit{syn.repl}\} && \{c_3, c_4\} \\
v_3 &= \mathbf{visit}\ 3 : \mathit{Tree}\ \{\mathit{inh.mini}, \mathit{syn.gath}, \mathit{syn.repl}\} && \{c_5, c_6\}
\end{aligned}
$$

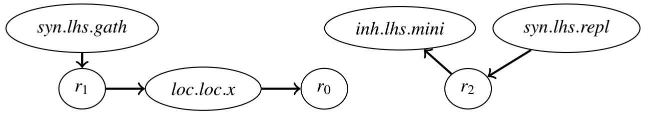
$$
\begin{aligned}
c_1 &= \mathbf{prod}\ \mathit{Leaf}\ \emptyset \\
c_3 &= \mathbf{prod}\ \mathit{Leaf}\ \emptyset \\
c_5 &= \mathbf{prod}\ \mathit{Leaf}\ \emptyset \\
c_2 &= \mathbf{prod}\ \mathit{Bin}\ \ \{\mathbf{sim}\ \{\mathbf{invoke}\ l\ 1, \mathbf{invoke}\ r\ 1\}\} \\
c_4 &= \mathbf{prod}\ \mathit{Bin}\ \ \{\mathbf{sim}\ \{\mathbf{invoke}\ l\ 2, \mathbf{invoke}\ r\ 2\}\} \\
c_6 &= \mathbf{prod}\ \mathit{Bin}\ \ \{\mathbf{sim}\ \{\mathbf{invoke}\ l\ 1, \mathbf{invoke}\ r\ 1\}, \mathbf{sim}\ \{\mathbf{invoke}\ l\ 2, \mathbf{invoke}\ r\ 2\}\}
\end{aligned}
$$

Figure 4.14: The contents of nodes and vertices of the visits graph of the example.

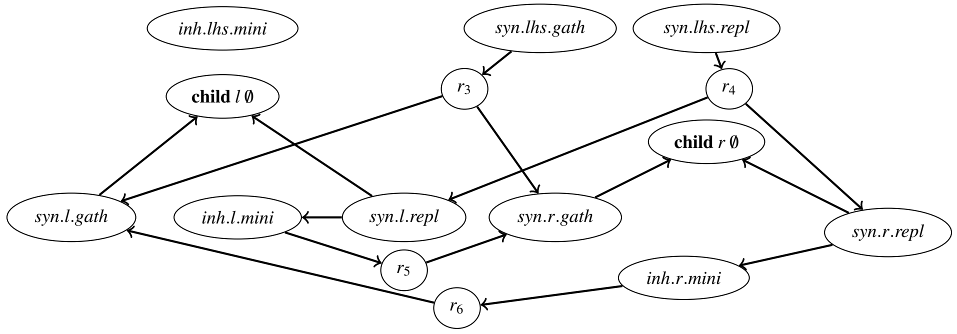
The relation *avail* states that a vertex *d* is available when all its dependencies are available, with the exception that only the inherited attributes *inh.lhs.y* are available if they are part of the configuration $\overline{a}$, and that synthesized attributes of visits are only available if there was a visit that computed them. Thus, *avail* gives us a notion of which vertices are scheduled in a VPDG in a given configuration.

Definition (Well-formed)**.** A visits-graph *g* is *well-formed* when it satisfies the invariants below. Additionally, the visits-graph is acyclic, and so are the VPDGs that are stored in the configurations. Also, in each edge of *g* or each vertex of *g* with some nonterminal *N*, there is exactly an entry $\langle \mathit{prop}\ P \ldots \rangle$ for each production *P* of *N*. We omit some straightforward structural invariants, such as that each visit-edge is annotated with the same nonterminal *N* as the nonterminal *N* of the two configurations it connects.

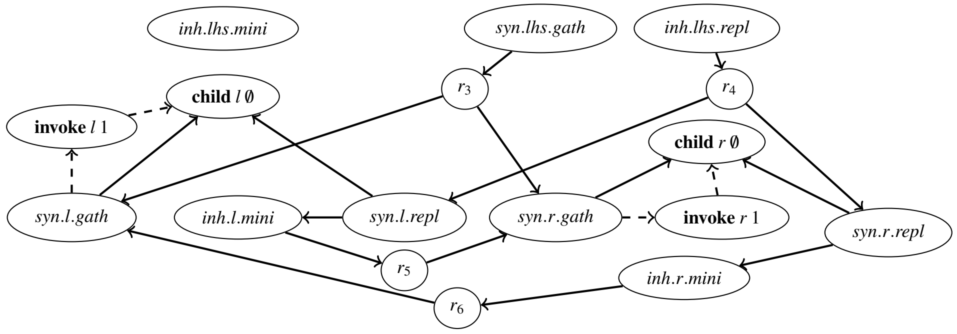
The main invariant is that all visits that are correctly represented in the graph, which is captured by the property in Figure 4.17. A visit-edge of nonterminal *N* requires a transformation of the VPDG for each production of *N*. The visit must precisely mention the new attributes $\overline{a}$ that are added to a_0 to form a_1. The sequence of simultaneous invocations $\overline{r}$ describes these transitions for the children. When applied to the VPDG g_0 of the old configuration, *embed* $\overline{a_1}\ g_0\ \overline{r}$ gives the unabstracted VPDG g_1 of the new configuration. Moreover, g_1 must be complete, which means that all the synthesized attributes *syn.lhs.y* in the new configura-



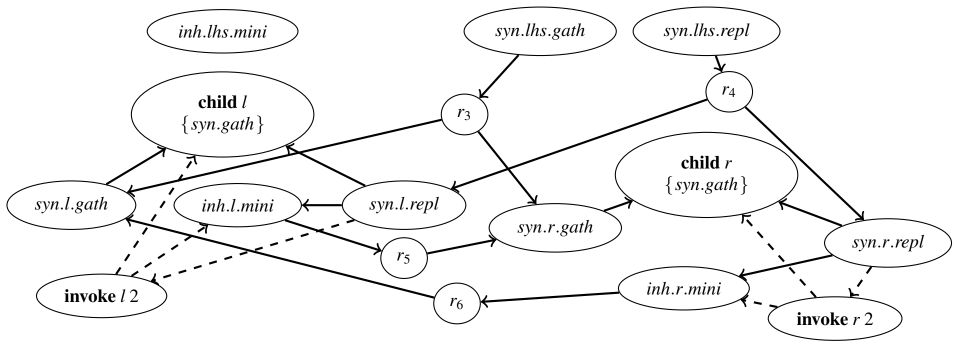
(a) VPDG g_1 of production *Leaf*, which has no children.



(b) VPDG g_2 of production *Bin*, which has children l and r.



(c) VPDG g_3 after visit v_1 in which it visited v_1 of the children.



(d) VPDG g_4 after visit v_2 or v_3 in which it visited v_2 and possibly v_1 of the children.

Figure 4.15: VPDGs of the visits graph example.

$$\begin{array}{lll}
\mathit{avail}\,\overline{a}\,g\,d & = & \mathit{all}\,(\mathit{avail}\,\overline{a}\,g)\,(\mathit{deps}\,g\,d) \wedge \mathit{except}\,\overline{a}\,g\,d \\
\mathit{except}\,\overline{a}\,g\,\langle\, \mathit{syn.x.y} \,\rangle & \mid & \langle \mathbf{visit}\,x\,i\rangle \in \mathit{deps}\,g\,\langle\, \mathit{syn.x.y}\,\rangle \\
 & \mid & \langle \mathbf{child}\,x\,\overline{a'}\,\rangle \in \mathit{deps}\,g\,\langle\, \mathit{syn.x.y}\rangle \wedge \langle \mathit{syn.x.y}\rangle \in \overline{a'} \\
\mathit{except}\,\overline{a}\,g\,\langle\, \mathit{inh.x.y} \,\rangle & \mid & \mathit{True} \\
\mathit{except}\,\overline{a}\,g\,\langle\, \mathit{syn.lhs.y} \,\rangle & \mid & \mathit{True} \\
\mathit{except}\,\overline{a}\,g\,\langle\, \mathit{inh.lhs.y} \,\rangle & \mid & \mathit{inh.y} \in \overline{a} \\
\mathit{except}\,\overline{a}\,g\,\langle\, \mathbf{rule}\,x \,\rangle & \mid & \mathit{True} \\
\mathit{except}\,\overline{a}\,g\,\langle\, \mathbf{child}\,x \,\rangle & \mid & \mathit{True} \\
\mathit{except}\,\overline{a}\,g\,\langle\, \mathbf{visit}\,x\,i \,\rangle & \mid & \mathit{True}
\end{array}$$

Figure 4.16: The definition of *avail*.

$$\begin{array}{l}
\langle\, \mathbf{conf}\,N\,\overline{a_0}\,\overline{p_0}\,;\, \mathbf{visit}\,i\!:\!N\,\overline{a}\,\overline{c}\,;\, \mathbf{conf}\,N\,\overline{a_1}\,\overline{p_1}\rangle \in \mathit{edges}\,\mathit{VG}_N = \\
\quad \mathit{all}_3\,(\mathit{transition}\,\overline{a_1}\,\overline{a})\,\overline{p_0}\,\overline{c}\,\overline{p_1} \;\wedge\; \overline{a_1} \equiv \overline{a_0} \uplus \overline{a} \\
\mathit{transition}\,\overline{a_1}\,\overline{a}\,(\mathbf{prod}\,P\,\overline{k_0}\,g_0)\,(\mathbf{prod}\,P\,\overline{r})\,(\mathbf{prod}\,P\,\overline{k_1}\,g_1) = \\
\quad \mathit{abstract}\,(\mathit{embed}\,\overline{a_1}\,g_0\,\overline{r}) \equiv g_1 \;\wedge\; \mathit{complete}\,\overline{a_1}\,\overline{a}\,g_1 \wedge \mathit{all}_2\,(\mathit{trchild}\,\overline{r})\,\overline{k_0}\,\overline{k_1} \\
\mathit{trchild}\,\overline{r}\,\langle\, \mathbf{child}\,x\!:\!N\,\overline{a}\,\rangle\,\langle \mathbf{child}\,x\!:\!N\,\overline{a'}\rangle \;=\; \overline{a'} \equiv \mathit{foldl}\,(\mathit{trsim}\,x)\,\overline{a}\,\overline{r} \\
\mathit{trsim}\,x\,\overline{a}\,\langle\, \mathbf{sim}\,\overline{m}\,\rangle \\
\quad \mid x \notin \mathit{children}\,\overline{m} \qquad\qquad\qquad\qquad\qquad\qquad\qquad = \; \emptyset \\
\quad \mid \langle\, \mathbf{invoke}\,x\,i\rangle \in_1 \overline{m} \;\wedge\; \langle s\,;\, \mathbf{visit}\,i\!:\!N\,\overline{a'}\,\overline{c}\,;\, s'\rangle \in \mathit{edges}\,\mathit{VG}_N \;=\; \overline{a} \uplus \overline{a'} \\
\mathit{children}\,\overline{m} \qquad\;\; = \; \{x \mid \langle \mathbf{invoke}\,x\,i\rangle \in \overline{m}\} \\
\mathit{complete}\,\overline{a_1}\,\overline{a}\,g_1 \;=\; \mathit{all}\,(\mathit{avail}\,\overline{a_1}\,g_1)\,\{k.\mathit{lhs}.x \mid k.x \in \overline{a}\}
\end{array}$$

Figure 4.17: Main invariant: are all visits represented.

tion must be available in g_1. Finally, the invocations $\overline{r}$ also describe the state transitions of the children. The relation *trchild* $\overline{r}$ k_0 k_1 relates the old configuration k_0 of a child to the new configuration k_1.

The relation *req* $\overline{a}$ *g d* in Figure 4.18 specifies if *d* is required for the synthesized attributes of $\overline{a}$ to be *available* in *g*. Note that the relation $d_1 \leftarrow^* d_2$ represents a possibly empty path between d_1 and d_2.

The function *embed* in Figure 4.18 applies the invocations of visits $\overline{r}$ to the VPDG *g*. A visit may only be invoked if its synthesized attributes are required, since our strategy schedules only the attributes that are needed. Moreover, the inherited attributes must be available.

The simultaneous invokes represent invocations that can be applied in any order. Since we test which inherited attributes are available according to g_0, the order in which these visits are applied does not affect the PDGs. This is because the synthesized attributes of the child are not available in g_0, and can thus not be used to make more inherited attributes of another

$$
\begin{aligned}
&\mathit{req}\,\overline{a}\,g\,d \;=\; d \in \mathit{reqs}\,\overline{a}\,g \\
&\mathit{reqs}\,\overline{a}\,g \;=\; \{d \mid \mathit{inh}.x \in \overline{a}, \mathit{syn}.y \in \overline{a}, (\mathit{inh}.\mathit{lhs}.x \leftarrow^{*} d \leftarrow^{*} \mathit{syn}.\mathit{lhs}.y) \in g\}
\end{aligned}
$$

$$
\begin{aligned}
&\mathit{embed}\,\overline{a}\,g \quad \langle\,\overline{r}\,\rangle \qquad = \mathit{foldl}\,(\mathit{embed}\,\overline{a}) \quad g\,\overline{r} \\
&\mathit{embed}\,\overline{a}\,g \quad \langle\,\mathbf{sim}\,\overline{m}\,\rangle \;= \mathit{foldl}\,(\mathit{embed}\,\overline{a}\,g)\;g\,\overline{r} \\
&\mathit{embed}\,\overline{a}\,g_0\;g\,\langle\,\mathbf{invoke}\,x\,i\,\rangle \\
&\quad \mid \mathit{all}\,(\mathit{avail}\,g_0\,\overline{a})\;\mathit{inhs} \wedge \mathit{all}\,(\mathit{req}\,g_0\,\overline{a})\;\mathit{syns} \\
&\quad = [\![\mathbf{visit}\;i\;\overline{a'}]\!]_{\mathsf{vert}\,(x)} \uplus [\![\mathbf{visit}\;i\;\overline{a'}]\!]_{\mathsf{edge}\,(x)} \uplus g \\
&\quad \mathbf{where} \\
&\qquad \langle\, s;\,\mathbf{visit}\;i : N\;\overline{a'}\;\overline{c}\,;\,s'\,\rangle \in \mathit{edges}\;\mathit{VG}_N \\
&\qquad \mathit{inhs} = \{\mathit{inh}.x.y \mid \mathit{inh}.y \in \overline{a'}\,\} \\
&\qquad \mathit{syns} = \{\mathit{syn}.x.y \mid \mathit{syn}.y \in \overline{a'}\,\}
\end{aligned}
$$

Figure 4.18: The relations *req* and *embed*.

child available. This property allows us to determine visits graphs in a stable way. Some modifications are possible to this approach to schedule attributes less eagerly.

Further, for each nonterminal N, there must be a vertex in the visits graph with an empty configuration. An empty configuration consists of an empty configuration for the children of each production. In addition, for the root symbols of the grammar with a non-empty set of attributes, the configuration with all attributes defined must be part of the visits graph. The configurations of the children do not have to be fully defined. Also, there must be an appropriate edge between these configurations.

Properties. The invariants imposed on the visits graph ensure a number of properties for which we sketch proofs.

The visits graph is acyclic. The destination configuration connected by an edge is larger than the edge's source configuration, because the set of attributes of an edge is not empty. Edges thus connect distinct configurations. The configurations connected by edges form an ascending chain, and can thus not be cyclic. □

The VPDGs are acyclic. The VPDGs of the initial configurations are by definition acyclic. The transformation induced by a visit adds visit-vertices, although these keep the VPDG acyclic. A vertex that is available requires its (indirect) dependencies to be available. Thus, an available vertex cannot depend on an unavailable vertex. A visit-vertex connects available vertices to unavailable vertices of synthesized attributes of a child. To form a cycle, such an available vertex must depend (indirectly) on one of the synthesized attributes. Since such a vertex is unavailable, a cycle is not possible. □

Consistent visits to children. The edges of the NDG of a child forms the edges between the child's inherited and synthesized attributes in the VPDG of a production that contains the child. Suppose that we visit child x with $\overline{a}$ as the attributes of the visit. As induction hypothesis, we assume that the available inherited and synthesized attributes of x are part of a consistent visit sequence. To prove that the visit sequence with the visit added is consistent, the definition requires us to consider four cases. In each case, one of the attributes is an element of $\overline{a}$.

As the first case, given an attribute *inh.x.y* and an attribute *syn.x.z*, suppose that $\mathit{inh.y} \leftarrow^{+} \mathit{syn.z}$ in the NDG of x, and $\mathit{syn.x.z} \in \overline{a}$. Consequently, vertex *syn.x.z* is unavailable. Then $\mathit{inh.x.y} \leftarrow \mathit{syn.x.z}$ in the VPDG, which means that *inh.x.y* is available. Hence, *syn.x.z* must be in the same or later visit as *inh.x.y*. Proofs for the remaining three cases are similar. □

Consistent states. When two potential edges in the visits graph converge, they result in the same state. Clearly, when two edges converge, the edges have an equivalent set of attributes $\overline{a_1}$ as destination state, otherwise the edges would not converge. Then, for both a VPDG g_1 of one edge, and a VPDG g_2 for the same production of the other edge, the *reqs* sets for $\overline{a_1}$ are equivalent. The attributes of a child's state are exactly those in the *reqs* set, hence the states of the children are equivalent. □

Dependencies between attributes of the same kind. A synthesized attribute *syn.x.y* can depend on a synthesized attribute *syn.x.z* and still be computed as part of the same visit when the inherited attributes both *syn.x.y* and *syn.x.z* depend on are available. Similarly, two inherited attributes can both be passed simultaneously to a child when they are both available. An inherited attribute that depends on a synthesized attribute of the same child, however, cannot be scheduled to the same visit. □

Construction. The size of the visits graph is in both the worst and average case exponential in the number of attributes. However, we normally need only a small portion of this graph. In the remainder of this chapter, we call the visits graph of a program a slice of the graph that we inferred from the program.

To incrementally construct this slice, we distinguish *partial* and final vertices and edges. A partial vertex contains the configuration $\overline{a}$, but not the administration for the productions. A final vertex does contain this information. Similarly, a partial vertex contains only the visit identifier i and the attributes $\overline{a}$.

The partial vertices and edges represent information of the graph that needs to be in the graph, but what we did not compute yet. The final vertices and edges represent already computed parts of the graph. For example, the initial vertices with an empty state are final vertices. For the root symbols, we insert a partial edge and partial destination vertex to the graph. As algorithm, we repeatedly take a partial edge with a final source vertex, and perform scheduling to turn it into a final edge, and consequently the destination vertex into a final vertex. Scheduling may result in new edges and vertices being added to the graph.

The algorithm that performs scheduling for final edges computes the information for the edge as a function of the source state and the attributes $\overline{a}$ of the edge. This ensures that the order in which we consider pending edges does not affect the resulting visits graph. Moreover, the visits graph is finite, thus if the computation for each pending edge is finite, then so is the whole computation.

As a given, the input to the algorithm is a set of attributes $\overline{a}$ of the edge, and the VPDGs of the productions of the previous configuration. To compute the remaining administration of the edge, we infer the visits to the children. The *embed* relation tells us how. We first determine which vertices are in the *reqs* set. Then we repeatedly which of the attributes of the children are required and ready to be scheduled. Those form one group of simultaneous visits. For each visit, we possibly add a pending edge and a pending vertex to the visits graph, if such vertices are not yet in the graph. This process terminates because the VPDG is acyclic. With each iteration, there is either at least one vertex that can be scheduled, or we are done.

The compilation of the largest AG of UUAG takes less than a minute using a slightly optimized version of the algorithm as sketched above. To improve the performance, the construction of the graph is relatively straightforward to parallelize. The processing of each edge is independent and can be done in parallel. Most of the shared state is read-only. Only updates on the state need to be properly synchronized, but these updates happen relatively infrequent, and there is likely little contention. The construction of the graph is thus likely to scale very well. Also, the process may be done incrementally. When a change in the AG does not affect the PDGs of a nonterminal, the graphs constructed for that nonterminal so far may be reused as initial visits graph. However, the implementation is performant enough to be used in practice, even without such optimizations.

Remarks. Both the construction of the PDGs and the visits graph is a whole-program analysis of the sources related to an evaluation algorithm of a compiler for a particular tree (e.g. one stage in the compiler pipeline). The requirement that the sources related to one algorithm must be analyzed as a whole is usually not a problem in large compiler implementations, because individual algorithms are usually monolithic. When plugins are concerned, such plugins are typically a separate stage, and would thus be compiled independently. However, it is in theory possible to defer the computations of the visits graph to load-time, although then the composition of rules must also be determined at load-time, using meta programming facilities such as Template Haskell.

There are various customizations posible in the construction on the visits graph which may have severe consequences for the structure of the graph. The structure of the graph does not influence the outcome of the attribute evaluation, but may affect execution time.

At the moment of writing, we are still gathering empirical data regarding performance. In earlier measurements on UHC, the difference between on-demand (lazy) and eager (strict) attribute evaluation turned out to be insignificant. This may be due to the use of strictness annotations and DeepSeq in combination with the sequentialization of computations due to unifications on a chained substitution attribute. However, in earlier measurements with the AG implementation of the editor Proxima [Schrage and Jeuring, 2004], execution time almost halved when eager AG evaluation was used. Note that with the rise of multi-core computing, the effects of visit sequences on parallel behavior may still be significant [Kuiper and Swierstra, 1990, Klaiber and Gokhale, 1992, Wang and Ye, 1991].

4.6 Optimizations

The shape of the graph may have an impact on the performance of the generated code. In our approach, visits are likely to be small and independent, which is beneficial for parallelism and incremental evaluation. However, when the visits graph is huge, code size becomes a more pressing issue. The visits graph of the largest AG in the UHC project features about 10,000 configurations. Hence, we require measures to limit the size of this graph.

Subsumption. When the graph is huge, there are many contexts that are similar, yet differ in one or more attributes because such an attribute was not needed or only needed later. Such a small difference can easily cause many visits to be needed in the graph.

An edge i from a configuration $\overline{a}$ with attributes $\overline{inh.y}$ and $\overline{syn.z}$ *subsumes* an edge k from $\overline{a}$ with attributes $\overline{inh.p}$ and $\overline{syn.q}$ when $\overline{syn.y} \subseteq \overline{inh.p}$ and $\overline{syn.q} \subseteq \overline{syn.q}$. When we are about to declare a visit, but it is subsumed by an already declared visit, we may use that visit instead. This approach can potentially save many contexts, at the expense of producing some results that are not needed yet. That is usually not a problem, because non-trivial computations of some node normally have dedicated inherited attributes, thus if one of these inherited attributes is needed then so is the non-trivial computation that computes the synthesized attribute.

A downside of the subsumption approach is that the order in which edges in the visits graph are considered may influence the outcome. The effects of such approaches requires further investigation.

We experimented with the strategy to determine the inherited attributes of a visit based on the synthesized attributes that are required, but determining the largest set of synthesized attributes that can be computed from the inherited attributes available so far. This strategy reduced the graph of UHC's largest AG to 1,500 nodes, at the slight expensive of returning one or two attributes more than is strictly necessary. Under the assumption that complex synthesized attributes are always dependent on their own set of inherited attributes, the additional cost is negligible.

Partial Kastens. Two other decisive factors are the number of attributes and productions, and sparse dependencies between attribues. The number of productions is typically fixed, but the number of attributes and the dependencies can be influenced. We can produce an algorithm for any consistent visit sequence. Consequently, a PDGs can be replaced with supergraphs as long as these remain acyclic.

In our experience, orderable AGs are absolutely non-circular (Section 1.3.4). If for one production there is a child x with nonterminal N that has a dependency of an attribute $inh.x.y$ on an attribute $syn.x.z$, then this dependency can also be imposed for all other children with nonterminal N. For such an AG, the approach of Kastens [1980] is applicable.

The approach of Kastens determines a total order on the attributes in the NDGs using a late-as-possible strategy. As mentioned earlier, this approach likely causes the graphs to become cyclic. By removing the edges between vertices of strongly connected components that are not in the original PDG, we obtain a non-cyclic supergraph of the original PDG. This supergraph has Kastens' algorithm partially applied, and is likely to have a significant lower number of possible contexts.

For large AGs, this appears to be the most effective step to limit the state explosion caused by many productions and attributes. It does not require manual intervention. On the other hand, it restricts the freedom in choosing smaller and independent visits.

Attribute elimination. In combination with copy rules, it is common practice to define attributes on nonterminals where they are actually unused, or only used in chains of copy rules. During the development of an AG, such attributes also show up because the AG is not finished, thus not all attributes are in use.

Superfluous attributes seem innocent, but actually make the scheduling harder. These attributes and their rules for such nonterminals are largely independent, thus easily lead to many contexts in the visits graph, because many ways to interleave them are allowed. Further, the Kastens' algorithm typically schedules them too early, which causes cycles in the dependency graphs, thus makes the above approach to reduce contexts less effective.

A combination of dead-code elimination and copy propagation [Nielson et al., 1999] can be used to eliminate superfluous attributes of a nonterminal. As an additional benefit, their removal may improve the performance of the application, because it prevents the trivial copying of attributes around the tree.

Similarly to the dependency analysis itself, many analyses for AGs can be specified as a recursive set of constraints. The common pattern is that some property of a nonterminal is determined by combining the properties of each production, which we call abstraction. This property of the nonterminal is then instantiated for each a child of the nonterminal. A fixpoint can then be computed starting with a fixed value for the nonterminals that are roots, and a bottom value for the other nonterminals.

In a similar way as shown in Section 4.4, we can define that an attribute is *live* if it is a dependency of a live attribute. For the root nonterminal, all synthesized attributes are live. Since this analysis is defined in terms of the attribute dependencies, the analysis can straightforwardly be defined in terms of the dependency graphs. As with dead-code elimination, the attributes and rules that are not live are removed from the grammar.

Similarly, a collection attribute [Magnusson et al., 2007] is *empty* if it is composed with either a copy rule or a monoid's append from empty attributes. In this case, we start with non-empty as bottom value, and do not treat start symbols of the grammar in a special way. Instead, a use rule that depends on no attributes becomes empty as value eventually.

Finally, the output attribute of a copy rule is a *copy* of an attribute x, where x either equals z when the input attribute y is a copy of another attribute z, or x equals y otherwise. This also applies to rules for which we can syntactically determine that its body is essentially the identity function, For other rules, the output attribute is not a copy of another attribute. Initially, each attribute is not a copy of any attribute. set of attributes. During abstraction, a synthesized attribute may be a copy of an inherited attribute if all productions agree on that attribute.

Given the results of the copy and empty analyses, a rule that depends on an attribute x that is a copy of y can substitute y for x. A rule that depends on an empty attribute can substitute the attribute with the empty value, and thus remove the dependency on the attribute. Finally, dead-code elimination cleans up the unused copies, unused empty attributes, and unused other attribues.

The updatable attributes provide some more options for extensions. When an attribute is *constant* or the attribute is *unique* [Hage et al., 2007], these attributes can be stored as a mutable structure in a global state, and replaced by a single attribute that stores a reference to that structure. These are analyses that are relatively straightforward to implement and exploit for BarrierAG, in contrast to general purpose programming languages, which have complications due to data types and higher-order functions.

4.7 Execution Plans and Generated Code

With each vertex in the visits graph, we uniquely associate an identifier j. For each edge of the visits graph, and for each of its productions, we construct an execution plan. The vertices of the VDPG that are required for the new state, but not required for the old state, are the edges that belong in the plan. In Chapter 3, we described how to derive a total order for the vertices. In this section, we simply assume that we take a topological sort of the graph.

Plan representation. Figure 4.19 shows the syntax of execution plans, and the execution plans for the production *Bin*. We organize the plans per production. For each production, it contains an execution plan for each edge in the visits graph. The rules are ordered, and visits to children are made explicit.

Definition (Intra-visit dependency)**.** With each vertex in the visits graph, we associate a set of *intra-visit* dependencies. These are values (denoted as descriptors d) in the state of a node that are potentially needed in later visits. Given a visits graph V, the set of descriptors contained in s for a production P is *intra P s* in Figure 4.20 (explained below).

The set of descriptors is determined by taking the descriptors introduced by by a visit, and those needed by rules of later visits. Also note that the graph g is the graph of the new state for *defs*, and the graph of the old state for *uses*. The difference is in the state of the child-nodes. Finally, the set of intra-visit dependencies of the initial state and each final state is empty.

From execution plans, we generate visit functions (see also Section 1.3.5). The process for individual visit functions is largely conventional (Section 2.1, and Section 5.3). However, the weaving of the visit functions is more complex, because we encode the visits graph, which is in general not a linear sequence of visits.

The visits graph specifies the possible states of an attributed tree, and models the visits that can be done on it. Moreover, it specifies which attributes are in which state, and which attributes are an input or output of a visit. In the generated code, an attributed tree is a value that represents a vertex in the visits graph. A visit function represents an edge. Figure 4.21 shows their types for the example. Below, we explain these types.

Types. For each vertex j of a nonterminal N in the visits graph, we generate a type $T\_N\_s_j$. This is the type for an attributed tree with nonterminal N and in configuration j. For each edge i of a nonterminal N in the visits graph, we generate a type $T\_N\_v_i$. This is the type

$$
\begin{array}{lll}
S & ::= \mathbf{sem}_{\Gamma}\; P : N\; \overline{c}\; \overline{q}\; j & \text{-- execution plans } \overline{q} \text{ of production } P\text{, and initial config } j \\
c & ::= \mathbf{conf}\; j\; \overline{a} & \text{-- description of a configuration } j \\
q & ::= s\,;\, s'\,;\, v & \text{-- edge between configurations } s \text{ and } s' \\
v & ::= \mathbf{visit}\; i : N\; \overline{a}\; \overline{r} & \text{-- visit combined with rules of the visit} \\
r & ::= \mathbf{child}\; x : N = f\; \overline{z^{\triangleright}} & \text{-- (higher order) child declaration} \\
 & \mid\; x : p\; \overline{z^{\triangleleft}} = f\; \overline{z^{\triangleright}} & \text{-- evaluation rule} \\
 & \mid\; \mathbf{sim}\; \overline{m} & \text{-- simultaneous invocations} \\
m & ::= \mathbf{invoke}\; x\; i : N & \text{-- child invocation} \\
s & ::= \mathbf{conf}\; j\; \overline{k} & \text{-- configuration of a node} \\
k & ::= \mathbf{child}\; x : N\; j & \text{-- configuration of child } x \\
j & & \text{-- configuration identifier}
\end{array}
$$

$$
\begin{array}{lll}
\multicolumn{3}{l}{\mathbf{sem}\; \{f_1 = \mathit{field\_l}, f_2 = \mathit{field\_r}, f_3 = \mathit{min}, f_4 = \mathit{Bin}\}\; \mathit{Bin} : \mathit{Tree}\; 1} \\
\multicolumn{3}{l}{\quad \{\mathbf{conf}\; 1\; \emptyset, \mathbf{conf}\; 2\; \{\mathit{syn.gath}\}, \mathbf{conf}\; 3\; \{\mathit{inh.mini}, \mathit{syn.gath}, \mathit{syn.repl}\}\}} \\
\multicolumn{3}{l}{\quad [\mathbf{conf}\; 1\; \{\mathbf{child}\; l : \mathit{Tree}\; 1, \mathbf{child}\; r : \mathit{Tree}\; 1\}} \\
\multicolumn{3}{l}{\quad ;\mathbf{conf}\; 2\; \{\mathbf{child}\; l : \mathit{Tree}\; 2, \mathbf{child}\; r : \mathit{Tree}\; 2\}} \\
\quad ;\mathbf{visit}\; 1 : \mathit{Tree} & \multicolumn{2}{l}{\mathbf{child}\; l : \mathit{Tree} = f_1} \\
 & \multicolumn{2}{l}{\mathbf{child}\; r : \mathit{Tree} = f_2} \\
 & \multicolumn{2}{l}{\mathbf{sim}\; \{\mathbf{invoke}\; l\; 1 : \mathit{Tree}, \mathbf{invoke}\; r\; 1 : \mathit{Tree}\}} \\
 & r_3 : \quad \mathit{id}\; \mathit{syn.lhs.gath}^{\bullet} & = f_3\; \mathit{syn.l.gath}^{\bullet}\; \mathit{syn.r.gath}^{\bullet} \\
\multicolumn{3}{l}{\quad ,\mathbf{conf}\; 2\; \{\mathbf{child}\; l : \mathit{Tree}\; 2, \mathbf{child}\; r : \mathit{Tree}\; 2\}} \\
\multicolumn{3}{l}{\quad ;\mathbf{conf}\; 3\; \{\mathbf{child}\; l : \mathit{Tree}\; 3, \mathbf{child}\; r : \mathit{Tree}\; 3\}} \\
\quad ;\mathbf{visit}\; 2 : \mathit{Tree} & r_5 : \quad \mathit{id}\; \mathit{inh.l.mini}^{\bullet} & = \mathit{id}\; \mathit{syn.r.gath}^{\bullet} \\
 & r_6 : \quad \mathit{id}\; \mathit{inh.r.mini}^{\bullet} & = \mathit{id}\; \mathit{syn.l.gath}^{\bullet} \\
 & \multicolumn{2}{l}{\mathbf{sim}\; \{\mathbf{invoke}\; l\; 2 : \mathit{Tree}, \mathbf{invoke}\; r\; 2 : \mathit{Tree}\}} \\
 & r_4 : \quad \mathit{id}\; \mathit{syn.lhs.repl}^{\bullet} & = f_4\; \mathit{syn.l.repl}^{\bullet}\; \mathit{syn.r.repl}^{\bullet} \\
\multicolumn{3}{l}{\quad ,\mathbf{conf}\; 1\; \{\mathbf{child}\; l : \mathit{Tree}\; 1, \mathbf{child}\; r : \mathit{Tree}\; 1\}} \\
\multicolumn{3}{l}{\quad ;\mathbf{conf}\; 3\; \{\mathbf{child}\; l : \mathit{Tree}\; 3, \mathbf{child}\; r : \mathit{Tree}\; 3\}} \\
\quad ;\mathbf{visit}\; 3 : \mathit{Tree} & \multicolumn{2}{l}{\mathbf{child}\; l : \mathit{Tree} = f_1} \\
 & \multicolumn{2}{l}{\mathbf{child}\; r : \mathit{Tree} = f_2} \\
 & \multicolumn{2}{l}{\mathbf{sim}\; \{\mathbf{invoke}\; l\; 1 : \mathit{Tree}, \mathbf{invoke}\; r\; 1 : \mathit{Tree}\}} \\
 & r_3 : \quad \mathit{id}\; \mathit{syn.lhs.gath}^{\bullet} & = f_3\; \mathit{syn.l.gath}^{\bullet}\; \mathit{syn.r.gath}^{\bullet} \\
 & r_5 : \quad \mathit{id}\; \mathit{inh.l.mini}^{\bullet} & = \mathit{id}\; \mathit{syn.r.gath}^{\bullet} \\
 & r_6 : \quad \mathit{id}\; \mathit{inh.r.mini}^{\bullet} & = \mathit{id}\; \mathit{syn.l.gath}^{\bullet} \\
 & \multicolumn{2}{l}{\mathbf{sim}\; \{\mathbf{invoke}\; l\; 2 : \mathit{Tree}, \mathbf{invoke}\; r\; 2 : \mathit{Tree}\}} \\
 & r_4 : \quad \mathit{id}\; \mathit{syn.lhs.repl}^{\bullet} & = f_4\; \mathit{syn.l.repl}^{\bullet}\; \mathit{syn.r.repl}^{\bullet}]
\end{array}
$$

Figure 4.19: Syntax of execution plans, and the plans of production *Bin*.

$$
\begin{array}{ll}
\multicolumn{2}{l}{\mathit{intra}\ P\ s = \cup\{\,(\mathit{uses}\ P\ s\ s' \cup \mathit{intra}\ P\ s') - \mathit{defs}\ P\ s\ s' \mid (s\,;s'\,;v) \in V\,\}} \\
\multicolumn{2}{l}{\mathit{uses}\ P\,\langle\,\mathbf{conf}\ N\ \overline{a_0}\ \overline{p_0}\,\rangle\ \langle\mathbf{conf}\ N\ \overline{a_1}\ \overline{p_1}\rangle \mid \langle\mathbf{prod}\ P\ \overline{k}\ g\rangle \in \overline{p_0} =} \\
\multicolumn{2}{l}{\quad \cup\,(\mathit{map}\ (\mathit{uses}\ P)\ (\mathit{reqs}\ \overline{a_1}\ g - \mathit{reqs}\ \overline{a_0}\ g))} \\
\mathit{uses}\ P\,\langle\,k.x.y\,\rangle & = \emptyset \\
\mathit{uses}\ P\,\langle\,\mathbf{child}\ x\ \overline{a}\,\rangle & \\
\quad \mid \mathit{abs}\ \overline{a} \equiv 0 & = \cup\{\,\mathit{map}\ \mathit{uses}\ \overline{z} \mid \langle\mathbf{child}\ x\!:\!N = f\ \overline{z^{\triangleright}}\rangle \in \mathit{rules}\ P\,\} \\
\quad \mid \mathit{length}\ \overline{a} > 0 & = \{\,\mathbf{child}\ x\ \overline{a}\,\} \\
\mathit{uses}\ P\,\langle\,\mathbf{rule}\ x\,\rangle & = \cup\{\,\overline{z_2} \cup \mathit{map}\ \mathit{uses}\ \overline{{z_1}^{\triangleleft}} \mid \langle x\!:\ p\ \overline{{z_1}^{\triangleleft}} = f\ \overline{{z_2}^{\triangleright}}\rangle \in \mathit{rules}\ P\,\} \\
\mathit{uses}\,\langle\,z^{\circ}\,\rangle & = \{\,z\,\} \\
\mathit{uses}\,\langle\,z^{\bullet}\,\rangle & = \emptyset \\
\mathit{uses}\,\langle\,z^{\times}\,\rangle & = \emptyset \\
\multicolumn{2}{l}{\mathit{defs}\ P\,\langle\,\mathbf{conf}\ N\ \overline{a_0}\ \overline{p_0}\,\rangle\ \langle\mathbf{conf}\ N\ \overline{a_1}\ \overline{p_1}\rangle \mid \langle\mathbf{prod}\ P\ \overline{k}\ g\rangle \in \overline{p_1} =} \\
\multicolumn{2}{l}{\quad \cup\,(\mathit{map}\ (\mathit{defs}\ P)\ (\mathit{reqs}\ \overline{a_1}\ g - \mathit{reqs}\ \overline{a_0}\ g))} \\
\mathit{defs}\ P\,\langle\,k.x.y\,\rangle & = \emptyset \\
\mathit{defs}\ P\,\langle\,\mathbf{child}\ x\ \overline{a}\,\rangle & = \{\,\mathbf{child}\ x\ \overline{a}\,\} \\
\mathit{defs}\ P\,\langle\,\mathbf{rule}\ x\,\rangle & = \cup\{\,\mathit{map}\ \mathit{defs}\ \overline{{z_1}^{\triangleleft}} \mid \langle x\!:\ p\ \overline{{z_1}^{\triangleleft}} = f\ \overline{{z_2}^{\triangleright}}\rangle \in \mathit{rules}\ P\,\} \\
\mathit{defs}\,\langle\,z^{\circ}\,\rangle & = \emptyset \\
\mathit{defs}\,\langle\,z^{\bullet}\,\rangle & = \{\,z\,\} \\
\mathit{defs}\,\langle\,z^{\times}\,\rangle & = \{\,z\,\}
\end{array}
$$

Figure 4.20: Intra-visit dependencies of nodes.

for a visit function that takes the tree from its source configuration $T\_N\_s_{[\![i]\!]_{\mathsf{source}}}$ to its target destination $T\_N\_s_{[\![i]\!]_{\mathsf{target}}}$.

We can apply one operation on an attributed tree. Given an *typed key* $K\_N\_s_j\ t$ and a tree $T\_N\_s_j$, the function $\mathit{inv}\_N\_s_j :: T\_N\_s_j \to K\_N\_s_j\ t \to t$ provides us with the visit function of type t:

$$\mathbf{data}\ T\_N\_s_j\ \mathbf{where}\ C\_N\_s_j :: \{\,\mathit{inv}\_N\_s_j :: \forall\,t.K\_N\_s_j\ t \to t\,\} \to T\_N\_s_j$$

The constructor $C\_N\_s_j$ is essentially a wrapper around the *inv* function.

The type t can be chosen by a parent by providing a key with this type. The key $K\_N\_s_j\ t$ is a type index. The child can inspect the key to discover which type is actually represented by t:

$$
\begin{array}{l}
\mathbf{data}\ K\_N\_s_j\ t\ \mathbf{where} \\
\quad K\_N\_v_{i_1} :: K\_N\_s\ T\_N\_v_{i_1} \\
\quad \ldots \\
\quad K\_N\_v_{i_n} :: K\_N\_s\ T\_N\_v_{i_n}
\end{array}
$$

For each outgoing edge i of j, $K\_N\_s_j$ contains a key $K\_N\_v_i$ that serves as evidence that t

```
type T_Tree = T_Tree_s1    -- initial configuration of the tree
      -- type of tree in a given state s (function from key to visit)
data T_Tree_s1 where C_Tree_s1 :: {inv_Tree_s1 :: ∀ t.K_Tree_s1 t → t} → T_Tree_s1
data T_Tree_s2 where C_Tree_s2 :: {inv_Tree_s2 :: ∀ t.K_Tree_s2 t → t} → T_Tree_s2
data T_Tree_s3 where C_Tree_s3 :: {inv_Tree_s3 :: ∀ t.K_Tree_s3 t → t} → T_Tree_s3
      -- type of a key, which identifies a visit v from state s
data K_Tree_s1 t where
  K_Tree_v1 :: K_Tree_s1 T_Tree_v1
  K_Tree_v3 :: K_Tree_s1 T_Tree_v3
data K_Tree_s2 t where
  K_Tree_v2 :: K_Tree_s2 T_Tree_v2
data K_Tree_s3 t where    -- empty data declaration
      -- type of a visit v, with continuation as the new state s
type T_Tree_v1 =          IO (Int,       T_Tree_s2)
type T_Tree_v2 = Int → IO (Tree,      T_Tree_s3)
type T_Tree_v3 = Int → IO (Int,Tree,T_Tree_s3)
```

Figure 4.21: Types of keys and semantic functions.

equals $T\_N\_v_i$. The type $K\_N\_s_j$ has no constructors when j has no outgoing edges in the visits graph. Indeed, a tree in such a configuration cannot be visited.

The type of a visit function $T\_N\_v_i$ is a function of values of the visit's inherited attributes to a computation of a tuple of values of the visit's synthesized attributes, and the new state of the tree. Since BarrierAG includes updatable attributes, the computation takes place in the IO monad:

$$\textbf{type}\ T\_N\_v_i = \tau_{inh_1} \to ... \to \tau_{inh_n} \to IO\ (\tau_{syn_1}, ..., \tau_{syn_m}, T\_N\_s_{[\![i]\!]_{\mathsf{target}}})$$

The type τ_{inh_k} is the type declared for the inherited attribute inh_k of nonterminal N, and τ_{syn_k} the type for the synthesized attribute syn_k.

Translation of semantics-blocks. Figure 4.22 gives the generated code for a production[3]. A sem-block is translated to a *node constructor* function st_j for each state j, which given values for the intra-dependencies of j, returns a node with this state, thus of type $T\_N\_s_j$. The node constructor st_1 constructs the initial state. Therefore it has an empty set of intra-dependencies and is thus represents the initial value of the node. In contrast, the node constructor st_2 takes the states of the live children as parameter, and values of the live attributes.

[3] The full code of the example can be downloaded from: `https://svn.science.uu.nl/repos/project.ruler.papers/archive/ExampleWarren.hs`. It is compilable with GHC version 6.12.3.

```
sem_Bin :: T_Tree → T_Tree → T_Tree
sem_Bin field_l field_r = st₁ where
  st₁ = let k₁ :: K_Tree_s₁ t → t
            k₁ K_Tree_v₁ = v₁
            k₁ K_Tree_v₃ = v₃
            k₁ _         = error "unreachable"
            v₁ :: T_Tree_v₁
            v₁           = do l₁ ← return f₁
                              r₁ ← return f₂
                              (l_gath, l₂) ← inv_Tree_s₁ l₁ K_Tree_v₁
                              (r_gath, r₂) ← inv_Tree_s₁ r₁ K_Tree_v₁
                              lhs_gath     ← return.id $ f₃ l_gath r_gath
                              return (lhs_gath, st₂ l₂ r₂ l_gath r_gath)
            v₃ :: T_Tree_v₃
            v₃ lhs_mini  = do l₁           ← return f₁
                              r₁           ← return f₂
                              (l_gath, l₂) ← inv_Tree_s₁ l₁ K_Tree_v₁
                              (r_gath, r₂) ← inv_Tree_s₁ r₁ K_Tree_v₁
                              lhs_gath     ← return.id $ f₃ l_gath r_gath
                              l_mini       ← return.id $ id r_gath
                              r_mini       ← return.id $ id l_gath
                              (l_repl, l₃) ← inv_Tree_s₂ l₂ K_Tree_v₂ l_mini
                              (r_repl, r₃) ← inv_Tree_s₂ r₂ K_Tree_v₂ r_mini
                              lhs_repl     ← return.id $ f₄ l_repl r_repl
                              return (lhs_gath, lhs_repl, st₃)
        in C_Tree_s₁ k₁
  st₂ l₂ r₂ l_gath r_gath
      = let k₂ :: K_Tree_s₂ t → t
            k₂ K_Tree_v₂ = v₂
            k₂ _         = error "unreachable"
            v₂ :: T_Tree_v₂
            v₂ lhs_mini  = do l_mini       ← return.id $ id r_gath
                              r_mini       ← return.id $ id l_gath
                              (l_repl, l₃) ← inv_Tree_s₂ l₂ K_Tree_v₂ l_mini
                              (r_repl, r₃) ← inv_Tree_s₂ r₂ K_Tree_v₂ r_mini
                              lhs_repl     ← return.id $ f₄ l_repl r_repl
                              return (lhs_repl, st₃)
        in C_Tree_s₂ k₂
  st₃ = let k₃ :: K_Tree_s₃ t → t
            k₃ _         = error "unreachable"
        in C_Tree_s₃ k₃
  f₁ = field_l  ;  f₂ = field_r  ;  f₃ = min  ;  f₄ = Bin
```

Figure 4.22: Generated code of the production *Bin*.

$$
\begin{array}{lcl}
\llbracket \mathbf{child}\; x : N = f\; \overline{z^{\triangleright}} \rrbracket_{\mathsf{gen}} & = & \llbracket x \rrbracket_{\mathsf{gen}\;(\mathsf{init}\;N)} \leftarrow \mathit{lift}_{|\overline{z^{\triangleright}}|}\; f\; \overline{\llbracket z^{\triangleright} \rrbracket_{\triangleright}} \\
\llbracket x : p\; \overline{z^{\triangleleft}} = f\; \overline{z^{\triangleright}} \rrbracket_{\mathsf{gen}} & = & \overline{y} \leftarrow \mathit{lift}_{|\overline{z^{\triangleright}}|}\; f\; \overline{\llbracket z^{\triangleright} \rrbracket_{\triangleright}} \\
 & & \overline{\llbracket z^{\triangleleft} \rrbracket_{\triangleleft\, y}} \qquad \mathbf{where}\; \overline{y}\; \mathit{fresh} \\
\llbracket \mathbf{invoke}\; x\; i : N \rrbracket_{\mathsf{gen}} & = & \mathbf{let}\; \mathit{vis} = \mathit{invoke}_{s_{\llbracket i \rrbracket_{\mathsf{source}}}}\; \llbracket x \rrbracket_{\mathsf{source}\;(i)}\; (K\_N\_\; v_i) \\
 & & (\llbracket x \rrbracket_{\mathsf{syn}\;(i)}, \llbracket x \rrbracket_{\mathsf{target}\;i}) \leftarrow \mathit{vis}\; \llbracket x \rrbracket_{\mathsf{inh}\;(i)} \\
\llbracket x \rrbracket_{\mathsf{inh}\;(i)} & = & \{\, \llbracket \mathit{inh.x.y} \rrbracket_{\mathsf{gen}} \mid \mathit{inh.y} \leftarrow a_{N_i} \,\} \\
\llbracket x \rrbracket_{\mathsf{syn}\;(i)} & = & \{\, \llbracket \mathit{syn.x.y} \rrbracket_{\mathsf{gen}} \mid \mathit{syn.y} \leftarrow a_{N_i} \,\} \\
\llbracket z^{\bullet} \rrbracket_{\triangleleft\, y} & = & \mathbf{let}\; \llbracket z \rrbracket_{\mathsf{gen}} = y \\
\llbracket z^{\circ} \rrbracket_{\triangleleft\, y} & = & \mathit{writeIORef}\; \llbracket z \rrbracket_{\mathsf{gen}}\; y \\
\llbracket z^{\times} \rrbracket_{\triangleleft\, y} & = & \llbracket z \rrbracket_{\mathsf{gen}} \leftarrow \mathit{newIORef}\; y \\
\llbracket z^{\bullet} \rrbracket_{\triangleright} & = & \mathit{return}\; \llbracket z \rrbracket_{\mathsf{gen}} \\
\llbracket z^{\circ} \rrbracket_{\triangleright} & = & \mathit{readIORef}\; \llbracket z \rrbracket_{\mathsf{gen}} \\
\llbracket h.c.x \rrbracket_{\mathsf{gen}} & = & h\_c\_x \\
\llbracket x \rrbracket_{\mathsf{gen}\;(j)} & = & x\_j
\end{array}
$$

Figure 4.23: Translation scheme for rules in the execution plan.

The body of a node constructor is a wrapper around a function k_j, which returns the visit function v_i given the key $K\_N\_\,v_i$ that identifies one of the outgoing edges of j. For each visit i that is a successor of configuration j, we generate a visit function v_i. The visit function v_i takes values for the inherited attributes as parameter that are declared on edge i. It returns a computation that gives a tuple of the synthesized attributes that are declared on edge i. In addition , it returns the result state of the node by applying the constructor for the next state to the values of that state's intra dependencies.

Figure 4.23 shows a straightforward translation of the rules in the execution plan. We assume that barrier attributes and dependency rules are stripped from the execution plan and the administration after the visits graph has been constructed.

For each visit to a child we pass as additional parameter the appropriate key to the invoke-function of the child's state, which results in the visit function *vis*. The function *vis* subsequently takes the inherited attributes as parameters.

For an attribute occurrence z at an input position, either the transcribed identifier $\llbracket z \rrbracket_{\mathsf{gen}}$ is z's value, or is a reference that can be read from to provide z's value. In a similar way, an attribute occurrence z at an output position is either stored as the transcribed identifier $\llbracket z \rrbracket_{\mathsf{gen}}$, or written to the reference under that name.

Remarks. A nice property of the translation is that we explicitly declare the types of the visit functions with type signatures. Since the types of the functions f are monomorphic, type errors are usually to be reported in functions f, which are defined in the actual source code, so that type errors can be related back to the original locations in the source file. We do not need to know the types of local attributes. Also, when we allow type variables in the types, then these can be supported in our scheme using scoped type variables.

The translation can be optimized in various ways. When a configuration does not have outgoing edges, a visit to a child that ends in this configuration does not need to construct nor return the new state of the child, as the child cannot be visited anymore, and the state is actually empty according to the definition of *intra*. When a configuration has one outgoing edge, then the selection of a visit via a key is not needed. Also, when the visits graph for a nonterminal is a tree, a single key that identifies the path in the tree can be given to the child when it is created, instead of one segment of the path for each visit.

Instead of relying on Haskell to construct closures, the node constructors can use an untyped, updatable array instead. With conventional techniques from register scheduling, the state can be represented such that needless copying of states is avoided during visits, which makes visits cheap. In this thesis, we do not venture down this path, and rely on Haskell to handle closures. A compiler that employes uniqueness and usage analysis [de Vries et al., 2007, Hage et al., 2007] can apply this optimization transparently.

When a simultaneous invocation group contains more than one visit, we can use *forkIO* to allow the Haskell runtime to evaluate the visits in parallel, since these do not have common dependencies.

4.8 Generalization to Phases

We use BarrierAG as a host language to describe phases. A nonterminal declares a set of phases. Attributes may be associated uniquely to a phase. A phase corresponds to one or more implicit visits. Since a visit describes the smallest unit of evaluation of a node in the tree, a phase describes a larger unit of evaluation. It allows us to express properties of chunks of AG evaluation, without resorting to the low level details of visits.

The following example is a possible declaration of phases for nonterminal *Tree*. The indentation determines which attribute is declared in which phase. The scope of the phase declaration ends before a keyword at the same indentation level:

```
itf Tree                 -- declaration of attributes and phases of Tree
  syn gath   :: Int      -- attribute not assigned to a phase
  phase distribute       -- declaration of a phase distribute
    inh mini :: Int      -- attribute mini in phase distribute
  phase transform        -- declaration of a phase transform
    syn repl :: Tree     -- attribute repl in phase transform
```

These phase declarations are unordered, and can be specified in any order.

Figure 4.24 shows the notation for phase interfaces. There is a high similarity with the notation for visit interfaces of Chapter 3. The main difference is that we may specify multiple blocks of subsequent phases in a phase-block. The lexical nesting of phase declarations (on a nonterminal) and phase blocks (in a sem-block) impose a partial order on phases. Also order-rules in combination with **begin** *lhs* ρ and **end** *lhs* ρ imposes a partial order. A phase-block

$I ::= \textbf{itf}\ N\ \overline{q}$	-- a set of declaration of a phase-interface
$q ::= a$	-- toplevel attribute decl (not associated with a phase)
$\mid\ \textbf{phase}\ \rho\ \overline{q}$	-- phase with attribute declarations
$d ::= \ldots$	-- descriptors extended with phases
$\mid\ \textbf{begin}\ x\ \rho$	-- begin of a phase ρ of child x
$\mid\ \textbf{end}\ x\ \rho$	-- end of a phase ρ of child x
$s ::= \textbf{sem}\ N\ \textbf{prod}\ P\ \overline{r}\ \overline{t}$	-- common rules r and a set of phase blocks $\overline{t}$
$t ::= \textbf{phase}\ \rho\ \overline{r}\ \overline{t}$	-- common rules r and subsequent phases $\overline{t}$
$r ::= \ldots$	-- AG rules
$\mid\ \textbf{invoke}\ \rho\ \textbf{of}\ \overline{c}\ z$	-- specifies invocation of phase ρ of $\overline{c}$ with strategy z

Figure 4.24: Notation for phase interfaces.

of phase ρ_2 that is nested in a phase block ρ_1 is evaluated either during the evaluation of ρ_1 when there is a constraint **begin** *lhs* $\rho_2 \prec$ **end** *lhs* ρ_1, or after the evaluation of ρ otherwise.

Invoke-rules may be explicitly given or be implicit. An invoke-rule r with a phase ρ corresponds to one or more actual visits to a child x. The rule itself precedes the first of these visits, thus $r \prec \textbf{begin}\ x\ \rho$.

The lexical scope of a phase block, which is relevant for local attributes and default-rules, is only determined by the nesting of phase blocks in a production. Possibly more constraints on the order induced by other productions or order-rules are not taken into account for scoping. This technical detail ensures that we can determine the vertices of the PDGs before the dependency analysis takes place.

We essentially have two main evaluation algorithms to choose from: demand-driven evaluation and statically ordered evaluation. We may specify several properties of a phase, such as cyclic or acyclic, and pure or impure. Not all combinations are possible. For example, a cyclic phase must use on-demand evaluation, and may not be impure. Also, the properties impose constraints on rules in such a phase, or on rules that invoke such a phase. For example, an invoke-rule in a pure phase may not invoke an impure phase on a child. The scheduling of rules takes such constraints into account (Section 3.5.2). In a host language with lazy evaluation, the on-demand algorithm is a simplification of the eager algorithm, hence in this thesis we focus only on the latter.

Foundation. We express phases in terms of BarrierAG, which we sketch in Figure 4.25. A dependency rule $k.x \prec k.y$ on attr-blocks of N is syntactic sugar for rule $k.lhs.x \prec k.lhs.y$ in each production of N. For each phase ρ of nonterminal N, we introduce two barrier-attributes $begin_\rho$ and end_ρ. The attributes of the phase are enclosed by these barriers. There is one master phase N for each nonterminal N where all attributes and phases are enclosed by via dependencies on its barriers.

The lexical nesting of phases and rules induces additional order constraints, as sketched by

```
attr N   inh begin_N barrier      -- begin master phase for N
         syn end_N   barrier      -- end master phase for N
attr N   inh begin_ρ barrier      -- begin barrier for each phase ρ of N
         syn end_ρ   barrier      -- end barrier for each phase ρ of N
attr N   inh.begin_ρ ≺ syn.end_ρ    -- begin before end for each phase ρ
         inh.begin_N ≺ inh.begin_ρ  -- begin ρ after begin master phase
         syn.end_ρ   ≺ syn.end_N    -- end ρ before end master phase
attr N   inh.begin_ρ ≺ k.y          -- for each attribute k.y of phase ρ
         k.z         ≺ syn.end_ρ    -- for each attribute k.y of phase ρ
         inh.begin_ρ ≺ inh.begin_ρ' -- for each nested phase ρ'
         syn.end_ρ'  ≺ syn.end_ρ    -- for each nested phase ρ'
         inh.begin_N ≺ k.y          -- for each attribute k.y not in a phase
         k.z         ≺ syn.end_N    -- for each attribute k.y not in a phase
```

Figure 4.25: Sketch of a translation of phases to BarrierAG.

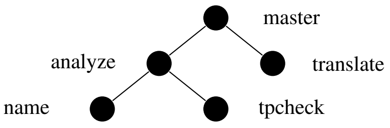


Figure 4.26: Phase nesting visualized as a tree.

the following example:

```
sem N $\overline{r_1}$          -- $inh.lhs.begin_N \prec \overline{r_1}$
  phase $\rho_1$        -- $inh.lhs.begin_N \prec inh.lhs.begin_{\rho_1}$
    $\overline{r_2}$            -- $inh.lhs.begin_{\rho_1} \prec \overline{r_1}$
    phase $\rho_2$      -- $inh.lhs.begin_{\rho_1} \prec inh.lhs.begin_{\rho_2}$
      $\overline{r_3}$          -- $inh.lhs.begin_{\rho_2} \prec \overline{r_3}$
```

With order-dependencies, and with syntax as demonstrated in Chapter 3, more dependencies between phases may be specified.

Further, we take the union of all constraints on phases of nonterminal N from each child with nonterminal N, and integrate these in the NDG of N. The PDGs must remain acyclic, otherwise the constraints on phases are inconsistent, which is considered a static error.

Phase nesting. A phase may be contained inside another phase. The parent of a node can invoke the contained phase of the node as part of the evaluation of the containing phase. This model introduces levels of granularity that allow for more concise specifications. We can visualize this model as a tree, which we show in Figure 4.26.

Definition (Nesting tree)**.** A *nesting tree* describes the nesting of phase declarations.

In a nesting tree, the nodes are phases. When a node ρ_2 is a child of a node ρ_1, ρ_1 is a nested phase of ρ_2. The above exemplary tree corresponds to the following phase interface:

```
itf Expr              -- defines the siblings for the root (master phase)
  phase analyze       -- nested phase in master phase
    phase name        -- nested phase in phase name
    phase tpcheck     -- nested phase in phase tpcheck
  phase translate     -- nested phase in master phase
```

The above phase interface describes that during the evaluation of the analyze phase of a child, the name phase of that child is invoked. Different strategies may be specified for the name phase than for the analyze phase.

The nesting tree can be inferred from the NDG. Given a nonterminal N, N's phase ρ_1 is a child of N's phase ρ_2, if either:

- $inh.begin_{\rho_2} \leftarrow^{+} inh.begin_{\rho_1}$ and $syn.end_{\rho_1} \leftarrow^{+} syn.end_{\rho_2}$. In this case, ρ_1 is fully enclosed by ρ_2.
- there exists a ρ_3 so that ρ_3 is a child of ρ_1, and ρ_3 is a child of ρ_2, with in the NDG $inh.begin_{\rho_2} \leftarrow^{+} inh.begin_{\rho_1}$ or $syn.end_{\rho_1} \leftarrow^{+} syn.end_{\rho_2}$. In this case, ρ_3 is both a child of ρ_1 and ρ_2, which we resolve by making ρ_1 a child of ρ_2.

The constraints form a directed graph per nonterminal, and can be solved with a fixpoint computation. If the resulting graph is not a tree, then either the constraints are inconsistent, or too few constraints were specified. Such a nesting tree leads to additional edges between the barriers of the NDG. If the PDGs become cyclic due to these additional edges, the constraints on phases were inconsistent. This approach infers a single nesting tree per nonterminal.

The inference of the nesting tree has the advantage that it becomes easier to compose phases. On the other hand, the nesting of phases is typically limited, and has a *purpose*, so it is only a slight burden to specify the nesting fully.

Overlap prevention. Sibling phases may not overlap, otherwise it is unclear which evaluation of a node corresponds to which phase. Two phases overlap if either $inh.x.begin_{\rho_2} \leftarrow^{+} inh.x.begin_{\rho_1}$ and $syn.x.end_{\rho_2} \leftarrow^{+} syn.x.end_{\rho_1}$, or the other way around. However, the dependencies on barriers do not guarantee this property. Given two sibling phases ρ_1 and ρ_2, we wish to express that either **end** $\rho_1 \prec$ **begin** ρ_2 or **end** $\rho_2 \prec$ **begin** ρ_1. Since siblings are not ordered, we do not know which of the two to take.

Attribute scheduling is actually a reduction of the ordering of phases. If we define phases such that each attribute is declared in a unique phase, then determining the order of phases is attribute scheduling. The order of phases may thus be dependent on context. Therefore, we take a similar approach as Section 4.5 and define the phases graph.

phases graph. The phases graph of a nonterminal is the phases graph of the master phase of the nonterminal. A phases graph is a DAG where each represents a phase, and an edge from a to b means that a comes before b in the sequence. A vertex is labelled with the phases graph of its children:

$g ::= \textbf{phase}\ \rho\ \overline{g}\ \overline{e}$ -- phases graph of phase ρ (initially the master phase)
$e ::= g \rightarrow g$ -- edges connect phases subgraphs

Such a graph g may have several sources and sinks. However, each path from source to sink must contain precisely all sibling phases, because each path corresponds to an ordered sequence of siblings in the nesting tree of the nonterminal.

Similar as the visits graph, the phases graph contains all consistent ways to interleave phases. An interleaving is consistent when it satisfies the partial order of the phases. In practice, there are only a few phases per nonterminal, with relatively dense constraints, thus the actual required portion of the graph contains little variety. Also, the problem is slightly easier compared to Section 4.5 because each path from source to sink is equally long, and the number of vertices on such a path is known beforehand.

We take a slightly different representation of the phases graph to stress the similarity with the visits graph. The vertices s of the phases graph represent an ordered sequence of the available $\textbf{inh}.\mathit{begin}_\rho$ attributes. An edge is annotated with the attribute that has become available:

$s ::= \overline{a}$ -- state of a vertex in the phases graph
$e ::= s\ \overrightarrow{a}\ s$ -- edge in the phases graph

Note that we can construct the nesting tree from such a sequence.

The purpose of the phases graph is to determine a small number of totally ordered nesting trees for a nonterminal. In a similar way as with visits, we determine a total order on the phases of children of a production given the total order on the phases of the production. For this analysis, we do not add visit-vertices to the PDGs. Instead, we add dependencies between attributes $\mathit{syn}.x.\mathit{end}_{\rho_1}$ and $\mathit{inh}.x.\mathit{begin}_{\rho_2}$ for children x and phases ρ_1 and ρ_2.

Recall that a vertex in a PDG is available when all its dependencies are available. In this situation, for a begin barrier to be ready, it must have been connected to an end barrier. Figure 4.27 gives the definition. Further, an end-barrier is required when the begin-barriers of the phase and its children are available. Sufficient child-edges must be added so that the (indirect) dependencies of a required end-barrier are available.

Definition (Selectable)**.** A begin-barrier is *selectable* when the following conditions are met.

- Its dependencies are available. This implies that the begin-barrier of its parent is available.

- If a begin-barrier of a sibling is available, then so is the end-barrier of that sibling. This ensures that the phases do not overlap.

Options for child-edges can thus be determined by taking the intersection between selectable and required begin-barriers. If this intersection is empty, then the dependencies on

$$
\begin{array}{lll}
\textit{avail}\ \overline{a}\ g\ d & = & \textit{all}\ (\textit{avail}\ \overline{a}\ g)\ (\textit{deps}\ g\ d) \wedge \textit{except}\ \overline{a}\ g\ d \\
\textit{except}\ \overline{a}\ g\ \langle\, \textit{inh.lhs.y} \,\rangle & \mid & \neg\ \textit{isBegin}\ \textit{inh.lhs.y} \vee \textit{inh.y} \in \overline{a} \\
\textit{except}\ \overline{a}\ g\ \langle\, \textit{inh.x.y} \,\rangle & \mid & \neg\ \textit{isBegin}\ \textit{inh.x.y} \vee (d \in \textit{deps}\ g\ \langle\, \textit{inh.x.y} \rangle \wedge \textit{isEnd}\ d) \\
\textit{except}\ \overline{a}\ g\ \langle\, \textit{syn.c.y} \,\rangle & \mid & \textit{True} \\
\textit{except}\ \overline{a}\ g\ \langle\, \textbf{rule}\ x \,\rangle & \mid & \textit{True} \\
\textit{except}\ \overline{a}\ g\ \langle\, \textbf{child}\ x \,\rangle & \mid & \textit{True} \\
\textit{except}\ \overline{a}\ g\ \langle\, \textbf{visit}\ x\ i \,\rangle & \mid & \textit{True} \\
\textit{isBegin}\ \langle\, \textit{k.x.begin}_{\rho} \,\rangle & & \\
\textit{isEnd}\ \ \langle\, \textit{k.x.end}_{\rho} \,\rangle & &
\end{array}
$$

Figure 4.27: The relation *avail* with begin and end barriers.

phases are inconsistent, i.e. forcing the barriers to overlap. If there are multiple options available, we prioritize based on some stable order, such as the order of appearance.

As initial solution, we start with only the begin-barrier of the master phase available for each nonterminal. For the root nonterminals, we construct as initial solution a full path by determining a last-as-possible order given the dependencies of the NDG, and prioritizing based on the stable order mentioned earlier.

Attribute scheduling. The set of paths from sink to source is in practice small in the phases graph. The amount of paths is the price to pay for not specifying a total order on phases. For each path in the phases graph, we determine the accompanying nesting tree. The set of these trees form the phase-interfaces of a nonterminal. For each child, we specify which phase-interface to use, using a type index for the child-rule.

With each phase-interface corresponds a specialized version of the NDG and PDGs. We also construct a separate visits graph per phase interface. Due to the additional dependencies imposed by the phases-interface, variability in the visits graph is reduced.

Moreover, on each nesting-tree, we can perform additional attribute scheduling. The attributes are restricted by a partial order, and this order may be strengthened if it does not lead to cycles in the NDG of the nesting tree. For example, we may schedule attributes to the latest possible phase, or avoid scheduling attributes to certain phases. Such heuristics are similar to those discussed for rule scheduling in Section 3.5.2.

Remarks. The generalization to phases ensures that our approach is a conservative extension of ordered attribute grammars. A conventional attribute grammar can be expressed by associating all attributes with one phase.

The inference of the phases graph is similar to the inference of the visits graph. It may be possible to combine the inference of both graphs. However, it is not immediately clear how to express heuristics, such as the avoidance of certain phases, in a combined approach.

$$
\begin{array}{llll}
\textbf{attr}\ \mathit{Tree} & \textbf{thr}\ \mathit{unq} :: \mathit{Int} & & \text{-- declaration of threaded attribute} \\
\textbf{sem}\ \mathit{Root} & & & \\
\quad \mid \mathit{Root} & \mathit{root.unq} = 1 & & \text{-- initial value of threaded attribute} \\
 & \mathit{loc.final} = \mathit{root.unq} & & \text{-- final value of threaded attribute} \\
\textbf{sem}\ \mathit{Tree} & & & \text{-- modifications to the threaded attribute} \\
\quad \mid \mathit{Leaf} & (\mathit{loc.myId}, \mathit{lhs.unq}) & = (\mathit{lhs.unq}, \mathit{lhs.unq}+1) & \mathit{lhs.unq} \diamond \mathit{lhs.unq} \\
\quad \mid \mathit{Bin} & \mathit{l.unq} & = \mathit{lhs.unq} & \mathit{lhs.unq} \diamond \mathit{l.unq} \\
 & \mathit{r.unq} & = \mathit{l.unq} & \mathit{l.unq} \diamond \mathit{r.unq} \\
 & \mathit{lhs.unq} & = \mathit{r.unq} & \mathit{r.unq} \diamond \mathit{lhs.unq}
\end{array}
$$

Figure 4.28: Example of commuting rules.

4.9 Commuting Rules

In this section, we present AGs with commuting rules, which are chained rules that can be reordered. These commuting rules provide an abstraction for the use of references in Section 4.3. Given an explicit ordering of rules, the composition of two rules is *commutative* when the two rules are *commutable*, which means that the rules may be swapped in the composition. Rule composition is a conditionally *commutative operator*. Commutable rules represent *commutable operations*. The swapping of rules models side effects, and commutativity facilitates reasoning about the safe use of side effects.

Syntax. We assume some conventional AG to start with, and show later how to encode the commutable rules in BarrierAG. Firstly, we introduce *threaded attributes*, which are a special form of chained attributes. Secondly, we introduce commuting rules, which specify a set $\overline{h}$ of commutable chains the rule participates in:

$$
\begin{array}{lll}
k ::= \ldots & & \text{-- attribute forms (e.g. } \textbf{inh} \text{ and } \textbf{syn}\text{)} \\
\quad \mid\ \textbf{thr} & & \text{-- threaded attribute} \\
r ::= \ldots & & \text{-- rules} \\
\quad \mid\ x\colon p\ \overline{z_1} = f\ \overline{z_2} & \overline{h} & \text{-- rule } x \text{ that commutes over } \overline{h} \\
h ::= z_1 \diamond z_2 & & \text{-- rule commutes with rules of } z_1 \text{ and } z_2
\end{array}
$$

The syntax $z_1 \diamond z_2$ specifies that the rule connects chains of z_1 and z_2. Attribute z_1 must be on an input position, and z_2 on an output position. Also, both must be threaded attributes, and their types must be the same.

The example in Figure 4.28 uses a threaded counter to demonstrate the commuting rules. Each value of *loc.myId* is unique, although it is not guaranteed that a right-sibling of a node has a higher *loc.myId* value. This would be the case if the rules were not commutable.

The example shows as the first rule of *Root* how to provide the initial value of a threaded attribute. This is a rule that defines a threaded attribute, but does not commute over it. Also,

sem $N \mid P$	-- k is a child of production P of nonterminal N
$\quad loc.k_x^{\mathsf{ref}} :: IORef\ \tau$	-- local attribute declarations with type
$\quad loc.k_x^{\mathsf{wr}}$ **barrier**	-- local barrier declaration
$\quad loc.k_x^{\mathsf{rd}}$ **barrier**	-- local barrier declaration
$\quad k.x_{\mathsf{ref}}{}^{\bullet} = loc.k_x^{\mathsf{ref}\bullet}$	-- copy as reference for child
$\quad k.x_{\mathsf{wr}} \prec loc.k_x^{\mathsf{wr}}$	-- restrain read barrier of child
$\quad k_x^{\mathsf{rd}} \quad \prec k.x_{\mathsf{rd}}$	-- restrain write barrier of child

Figure 4.29: Sketch of the encoding of commuting rules in BarrierAG.

the example shows as the second rule of *Root* how to obtain the final value of a threaded attribute, which is a rule that refers to the value of a threaded attribute, but does not commute over it. Finally, the rules of *Tree* are commutable rules.

As additional requirement, a rule that defines a threaded attribute *lhs.y* must mention *lhs.y* in its set of commuting chains $\overline{h}$. This restriction ensures that the threaded attribute can be represented as an inherited attribute.

The commuting chains $\overline{h}$ specifies in which chains a rule participates:

$$\begin{aligned} uses\ \overline{h} &= \{z_1 \mid \langle z_1 \diamond z_2 \rangle \in \overline{h}\} \\ defs\ \overline{h} &= \{z_2 \mid \langle z_1 \diamond z_2 \rangle \in \overline{h}\} \end{aligned}$$

A rule with $\overline{h}$ connects *uses* $\overline{h}$ to *defs* $\overline{h}$.

BarrierAG encoding. We use a BarrierAG encoding as a means to specify the implementation of commuting rules. We represent each threaded attribute **thr** $x :: \tau$ with three attributes in BarrierAG:

inh $x_{\mathsf{ref}} :: IORef\ \tau$	-- reference to the mutable state
syn x_{wr} **barrier**	-- all commutable updates before this barrier
inh x_{rd} **barrier**	-- all non-commutable updates after this barrier

Thus, a threaded attribute is a reference to a mutable state. We ensure in the encoding that rules only depend on the reference, which permits the reordering. As invariant, the write barrier of the child depends on all commuting rules of the child (and its subtree) that update the reference. All non-commuting rules that refer to the reference depend on the read barrier.

For each threaded attribute **thr** $y :: \tau$ of a child k, we introduce three local attributes, and rules to connect the local attributes with the attributes of the child. Figure 4.29 gives a sketch. Note that k_x^{ref} is the name of the attribute. The name of the child k is part of the name of the attribute.

When a rule refers to the threaded attribute of k, we actually let it refer to the local attributes, as we show below. For notational convenience, we also introduce these local attributes for threaded attributes of *lhs*:

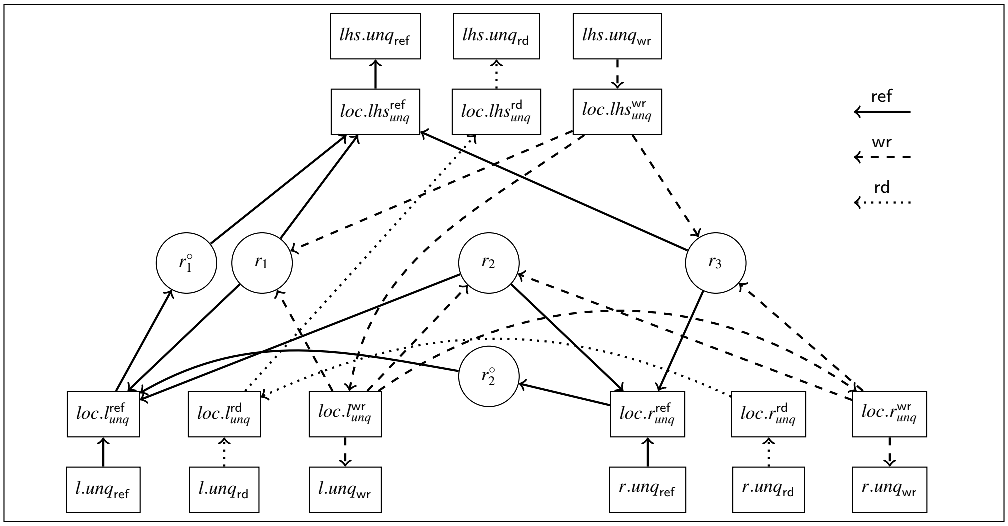

$lhs.unq_{\mathsf{ref}}$
$lhs.unq_{\mathsf{rd}}$
$lhs.unq_{\mathsf{wr}}$
$loc.lhs^{\mathsf{ref}}_{unq}$
$loc.lhs^{\mathsf{rd}}_{unq}$
$loc.lhs^{\mathsf{wr}}_{unq}$
ref
wr
rd
r_1°
r_1
r_2
r_3
r_2°
$loc.l^{\mathsf{ref}}_{unq}$
$loc.l^{\mathsf{rd}}_{unq}$
$loc.l^{\mathsf{wr}}_{unq}$
$loc.r^{\mathsf{ref}}_{unq}$
$loc.r^{\mathsf{rd}}_{unq}$
$loc.r^{\mathsf{wr}}_{unq}$
$l.unq_{\mathsf{ref}}$
$l.unq_{\mathsf{rd}}$
$l.unq_{\mathsf{wr}}$
$r.unq_{\mathsf{ref}}$
$r.unq_{\mathsf{rd}}$
$r.unq_{\mathsf{wr}}$

Figure 4.30: Attributes encoding the commutative rules of the production *Bin*.

$$\begin{array}{lll} loc.lhs_x^{\mathsf{ref}^\bullet} = lhs.x_{\mathsf{ref}}^{\bullet} & & \text{-- copy as reference} \\ loc.lhs_x^{\mathsf{wr}} \prec lhs.x_{\mathsf{wr}} & & \text{-- restrain read barrier} \\ k.lhs_{\mathsf{rd}} \quad \prec lhs_x^{\mathsf{rd}} & & \text{-- restrain write barrier} \end{array}$$

These rules for *lhs* are the contravariant version of the rules for children.

Non-commuting write. When a rule x with commuting chains $\overline{h}$ defines a threaded attribute $thr.k.y$, but $thr.k.y \notin defs\ \overline{h}$, rule x serves as *initializer* for the threaded attribute. In the encoding, the defining occurrence $thr.k.y$ in the left-hand side of rule x is replaced with $loc.k_y^{\mathsf{ref}^\times}$. Moreover, we add the following dependency rule:

$$loc.k_y^{\mathsf{wr}} \prec loc.k_y^{\mathsf{rd}} \quad \text{-- orders writes before reads}$$

Since k is the start of the chain, and given the invariants on the read and write barriers, this dependency rule ensures that commutable writes take place before non-commutable reads.

Non-commuting read. When a rule x with commuting chains $\overline{h}$ refers to a threaded attribute $thr.c.y$, but $thr.c.y \notin uses\ \overline{h}$, rule x reads the final value of the threaded attribute. Thus, we replace the occurrence $thr.c.y$ in the right-hand side of rule x with $c_y^{\mathsf{ref}^\circ}$, and add the following dependency:

$$loc.c_y^{\mathsf{rd}} \prec \textbf{rule}\ x \quad \text{-- the read depends on the read barrier}$$

The identifier c is either a child k or *lhs*.

Commuting read and write. When a rule x with commuting chains $\overline{h}$ refers to a threaded attribute $thr.c.y$, and $thr.c.y \in defs\ \overline{h}$, we replace the defining occurrence in the left-hand side of x with $c_y^{\mathsf{ref}^\circ}$. When $thr.c.y \in uses\ \overline{h}$, we replace the occurrence in the righthand side of x with $c_y^{\mathsf{ref}^\circ}$. Moreover, we insert a rule x° for each commuting chain, which copies the reference and thus links the chain. Also, we connect the read and write barriers:

$x^{\circ} : id\ k.y_2{}^{\bullet} = id\ c.y_1{}^{\bullet}$ -- for each $c.y_1 \diamond k.y_2 \in \overline{h}$ (copies the reference)
rule $x \prec loc.c^{\mathsf{wr}}_{y}$ -- for each attribute y in $\overline{h}$ (rule before write barrier)
$loc.c^{\mathsf{wr}}_{y_2} \prec loc.k^{\mathsf{wr}}_{y_1}$ -- for each $c.y_1 \diamond k.y_2 \in \overline{h}$ (connect write barriers)
$loc.c^{\mathsf{rd}}_{y_1} \prec loc.k^{\mathsf{rd}}_{y_2}$ -- for each $c.y_1 \diamond k.y_2 \in \overline{h}$ (connect read barriers)

When $lhs.y \in defs\ \overline{h}$, the read and write barrier are not connected, as it would create a cycle. Also, note the contravariant behavior between the read and write barrier.

The barrier attributes and dependency rules enforce the proper ordering of rules that use threaded attributes. Figure 4.30 demonstrates the encoding for the Bin-production. The boxes represent attributes. The circles represent the relevant rules. The barriers and their dependencies are not part of the generated code, and thus do not have a runtime overhead. As a consequence, we actually transformed a chained attribute into an inherited attribute with a reference to a mutable state.

Referential transparency. Referencial transparency is important for equational reasoning. For AGs, it is also important to ensure that the order of evaluation does not affect the result. In BarrierAG, rules that use updatable attributes break referential transparency. However, with commutable rules, we can establish a weaker version of referential transparency. When two rules commute, the actual values for the attributes that these rules define may be different, but in the context where these rules are defined, the final result, which abstracts from the values of the attributes, may still be equivalent to any ordering of the commutable rules.

The composition of rules, and the context of rules can be made explicit with arrow notation. A commuting rule $r_1 : (x_1, y_1) = f\ (x_0, y_0),\ x_0 \diamond x_1$ and a commuting rule $r_2 : (x_2, z_1) = g\ (x_1, z_0),\ x_1 \diamond x_2$ correspond respectively to the arrows $(x_1, y_1) \leftarrow f \prec (x_0, y_0)$ and $(x_2, z_1) \leftarrow g \prec (x_1, z_0)$. Section 1.3.9 shows that the composition of these rules can be expressed as an arrow:

```
proc (x0, y0, z0) → do
  (x1, y1) ← f ≺ (x0, y0)
  (x2, z1) ← g ≺ (x1, z0)
  returnA (x2, y1, z1)
```

Alternatively, when we reorder f and g, and rename the attributes, we obtain the following arrow:

```
proc (x0, y0, z0) → do
  (x1, z1) ← g ≺ (x0, z0)
  (x2, y1) ← f ≺ (x1, y0)
  returnA (x2, y1, z1)
```

This notion can straightforwardly be generalized to rules that commute over many attributes, or define and use many other attributes.

These two rules are *commutable* over attributes of x_0, x_1 and x_1, x_2 if their compositions are equivalent for a given rule context h, and $r_1 \not\prec r_2$:

$$h \left(\begin{array}{l} \textbf{proc}\ (x_0, y_0, z_0) \rightarrow \textbf{do} \\ \quad (x_1, y_1) \leftarrow f \prec (x_0, y_0) \\ \quad (x_2, z_1) \leftarrow g \prec (x_1, z_0) \\ \quad \textit{returnA}\ (x_2, y_1, z_1) \end{array} \right) \equiv h \left(\begin{array}{l} \textbf{proc}\ (x_0, y_0, z_0) \rightarrow \textbf{do} \\ \quad (x_1, z_1) \leftarrow g \prec (x_0, z_0) \\ \quad (x_2, y_1) \leftarrow f \prec (x_1, y_0) \\ \quad \textit{returnA}\ (x_2, y_1, z_1) \end{array} \right)$$

If there exists directly or indirectly a dependency between r_1 and r_2 then the rules may not commute. This is for example the case when r_1 defines a (non-threaded) attribute that is used by r_2, or because of a dependency rule.

The rule context h is an abstraction of the composition in which the composition of f and g is contained. For example, h can represent the composition of the rules of the entire tree, and thus the rule states that the end result of the computation is not affected. In practice, we take a more abstract notion of h. For example, the property that all *loc.myId* attributes have a unique value. In Section 2.2 we give some examples of the function h.

Remarks. To reason with commutable rules, we may need to make assumptions about the order of evaluation. This is, for example, the case when the values of the attributes are trace monoids [Diekert and Métivier, 1997]. Phases can be used for this purpose.

The identification of commuting rules may be relevant for the parallel and incremental evaluation of attribute grammars. Chained attributes sequentialize code, whereas commuting rules allow more interleaving. Similarly, during incremental evaluation, changes in a subtree that appears earlier in the evaluation may be lifted over a later subtree if these changes are visible in a threaded attribute.

Commutable rules can also be used to collect statistics or other runtime properties about the evaluation process, such that the attribute-dependencies of the collecting rules have only a minor influence on the evaluation process. For example, a count of the nodes of the tree traversed so far may be an indication of how much work has been done.

In case of type inference, substitutions may be represented as a threaded attribute, so that the threading of the substitution does not influence the order of evaluation. Traditionally, unification is only a commutable operation when all unifications succeed. By encoding the substitution as a graph structure, it is possible to make unification a commutable operation also in the case of a type conflict [Heeren, 2005].

4.10 Related Work

The ability to compose attribute grammars is an important benefit that attribute grammars offer. In fact, AGs are so easily composed that the compositions may accidentally become inconsistent, i.e. have attributes with a cyclic definition. Knuth [1968] proves that an AG is well-defined if and only if the dependency graphs of productions are cycle-free according to his refined algorithm (Knuth-2). When the dependency graphs are cycle-free according to Knuth's original algorithm (Knuth-1) then the AG is well-defined, but not necessarily the other way around. These are static properties of AGs that provide guarantees that the evaluation of an AG terminates.

Knuth-2 uses a dependency graph per production/child production combination, in contrast to a single dependency graph per production as the Knuth-1 approach uses. Knuth-2 leads to an approximate number of dependency graphs per production in the order (p^b), where p is the number of productions of the child nonterminals, and b is the number of children. In practice, e.g. for the let-production of a lambda calculus, p is rather large ($p > 10$), but b is typically small ($b \leqslant 2$). However, we usually define a fine granularity of nonterminals so that distinct dependency graphs per production/child production combination does not offer an advantage over a single graph per production.

In our experience, AGs are either necessarily cyclic, or are cycle-free in both Knuth-1 and Knuth-2. If the AG is not cycle-free with Knuth-1, then this is an indication that the AST does not have sufficient structure, which is not likely in strongly typed languages. In the first case, on-demand evaluation may still yield results for attributes, although it is the responsibility of the programmer to ensure this. In the second case, we know that an evaluation order exists, and can use a statically ordered evaluation algorithm. A statically ordered evaluation algorithm is likely to exhibit better time and space behavior, and actually permits minor assumptions about the evaluation order to be made.

Kastens [1980] presented an approach to infer a visit interface per nonterminal. Unfortunately, when using the Kastens approach, we often encounter cycles that are induced by the scheduling as resulting from Kastens' scheduling algorithm. These induced cycles hamper compositionality, because as remedy, we need to add artificial dependencies between attributes to the AG to control the scheduling. Also, the effect of scheduling is not visible in the original rules of the grammar, which makes such cycles very hard to understand and resolve.

The approach by Kennedy and Warren [1976] can find a solution, but may possibly result in an exponentially large solution. In practice the solution is not so large: In our experience, this approach works very well for small AGs. For large AGs, however, the exponential behavior may show up. To counter this behavior, we provide sufficient mechanisms to restrict the set of solutions.

4.11 Conclusion

This chapter demonstrates one of the great strengths of attribute grammars: the ability to statically analyse the grammar with abstract interpretations. As one of the main contributions of this chapter, we reformulated the approach of Kennedy and Warren [1976] so that it can be used to generate code for a strongly typed, purely functional host language. For absolutely non-circular AGs, this approach finds a way to order the rules statically.

By using such an approach, it is not needed to explicitly schedule attributes and rules, unless these need to be given special properties. We introduce the notion of phases for this. Since the ordering of attributes and rules can be inferred, we may omit such details from our specification. Consequently, the specification becomes more concise and easier to compose, which is beneficial from an engineering point of view.

The price that we pay is that such an analysis can only give an answer per context, such as a context dependent phase or visit interface, and a nonterminal may be in exponentially many contexts. A programmer typically has some evaluation order in mind, thus is usually able to specify a single order that works in all contexts, which may reduce the size of the generated code, compile time, and the execution time. On the other hand, if there is not a single order that works in all contexts, then a programmer is not likely to be able to keep track of all the possibilities manually. We presented phases as a means to specify knowledge about the order of evaluation, without going into the fine details, as in Chapter 3.

Also, for some problems, the order imposed by attribute dependencies may be too restrictive. We presented commuting rules as a means to loosen some restrictions. Commuting rules can be used when a number of rules form a chain, and the individual ordering of the rules is not relevant for the result. A typical example is a sequence of rules that provides unique numbers to nodes in the tree. When the requirement is only that each number is unique, the actual order in which numbers are handed out is does not invalidate that requirement.

As future work, more quantitative insight is needed about the effects of dependency analysis on the performance of the generated code. In earlier experiments, the impact seemed negligible, although the results differed from one AG to another. Also, we need more insight which heuristics have impact on the size of the visits graph. For example, it appeared that our approach in which we only compute what is needed resulted in less paths in the visits graph than the approach where we compute as much as possible. However, more quantitative evidence is required to draw such conclusions.

Another direction of future work is an extension of the Kennedy-Warren approach with support for cyclic AGs. Given a collection of cyclic PDGs, we can determine which attributes of the NDGs are mutually dependent. By grouping these attributes in a separate phase that uses lazy evaluation (for example) as evaluation algorithm, we can support a combination of a cyclic and non-cyclic AGs.

5 Derivation Tree Construction

Type inference on a program is the gradual process of constructing a *typing derivation*, which is a proof that relates the program to a type. During this process, inference algorithms analyze the intermediate states of the typing derivation to direct the construction of the proof.

Since a typing derivation is a decorated tree, we aim to use attribute grammars to implement type inference. In their present form, it is hard to express type inference in attribute grammars, because attributes are defined in terms of the final state of the decorated tree.

We present the language RulerCore, a conservative extension to ordered, higher-order attribute grammars, that permits both the structure and attributes of the tree to be defined based on intermediate states of the tree. We show that both iteration-based and constraint-based inference algorithms can be expressed straightforwardly in RulerCore.

5.1 Introduction

Attribute grammars (AGs) are traditionally used to specify the static semantics of programming languages [Knuth, 1968]. Moreover, when semantic rules of an AG are written in a general purpose programming language, the AG can be compiled into an (efficient) multi-visit tree walk algorithm that implements the specification [Kennedy and Warren, 1976, Kastens, 1980].

We implemented a substantial part of the Utrecht Haskell Compiler (UHC) [Dijkstra and Swierstra, 2004, Fokker and Swierstra, 2009] with attribute grammars using the UUAG system [Löh et al., 1998]. Haskell [Hudak et al., 1992] is a purely functional programming language, with an elaborate and expressive type system. We also compile our attribute grammars to Haskell. The ideas presented in this chapter, however, are not restricted to any particular language.

Attribute grammars benefit the implementation of a compiler for several reasons. Firstly, the evaluation order of semantic rules is determined automatically, unrelated to the order of appearance. Rules may be written separately from each other, and grouped by aspect, which makes attribute grammars highly composable [Viera et al., 2009, Saraiva, 2002]. Secondly, semantic rules for idiomatic tree traversals (such as: topdown, bottom-up, and in-order) can be inferred automatically, thus allowing for concise specifications. These advantages play an important role in the UHC project [Dijkstra et al., 2009].

Two essential components of UHC's type inferencer, polymorphic unification and context reduction, would benefit from an AG-based implementation. For example, when polymorphic unification is defined as an AG, many of its required attributes can be automatically provided by the AGs of expressions and declarations. However, we implemented these components directly in Haskell, because it is not obvious how to express these as an attribute grammar.

These two components present two challenges to attribute grammars. Firstly, the grammar needs to produce typing derivations. The structure of such a derivation depends on what is known about types, and this information gradually becomes available during inference, typically as a result of unifications. This requires a mixture of tree construction and attribute evaluation, which are normally separate tasks if one takes an AG view. Secondly, the construction of a proof of a subgoal may need to be postponed when it depends on a type that is not known yet. After more of the structure of the proof is determined, the type may become known and the postponed construction can continue.

An evaluator for an attribute grammar starts from a given tree (usually constructed by the parser), and evaluates the attributes using a fixed algorithm. We present AG extensions to customize the algorithms, without loosing the advantages that AGs offer. More precisely, our contributions are:

- We present the language RulerCore, a conservative extension of ordered attribute grammars. It has three concepts to deal with the above challenges:
  - We exploit the notion of visits to the tree. In each visit, some attributes are computed, as we explained in Chapter 3. Visits can be done iteratively. The number of iterations can be specified based on the values of attributes.
  - We define abstract grammars on the structure of typing derivations. Productions are chosen based on the values of attributes. Moreover, we present *clauses*, which allows the choice of a production to be refined per visit.
  - Derivation trees are first class values in RulerCore. They can be passed around as attributes, and can be inspected by visiting them.

  In Section 5.3, we define a denotational semantics for RulerCore via a translation to Haskell.

- In Chapter 3, we gave an introduction to RulerCore. In this chapter, we show how RulerCore can be used to express type inference. Section 5.2 presents an extensive example that motivates the design of RulerCore.

- An implementation of RulerCore is available at: `https://svn.science.uu.nl/repos/project.ruler.papers/archive/ruler-core-1.0.tar.gz`. It includes several examples. The implementation supports both statically ordered and demand driven evaluation of attributes.

- We compare our approach with other attribute grammar approaches (Section 5.4) as a further motivation for the need for RulerCore's extensions.

5.2 Motivation

In this section, we show how to implement a small compiler for an example language we named SHADOW, written with attribute grammars using RulerCore. The implementation of SHADOW poses exactly those challenges mentioned in the previous section, while being small enough to fit in this chapter.

5.2.1 Example: the Shadow-language

We take for SHADOW the simply-typed lambda calculus, with two small modifications: we annotate bindings with a unique label u (i.e. $\lambda x^u.$), and allow identifiers to refer to *shadowed* bindings. For example, in the term $\lambda x^{u_1}.\lambda x^{u_2}.f\ x$, the expression x normally refers to the binding annotated with u_2. However, if the expression cannot be typed with that binding, we allow x to refer to the shadowed binding annotated with u_1 instead, if this interpretation would be well-typed[1]. The interpretation of this expression is thus by default $\lambda u_1.\lambda u_2.f\ u_2$, but under some conditions (defined more precisely further below) it may be $\lambda u_1.\lambda u_2.f\ u_1$.

The compiler for SHADOW takes an expression (of type *ExprS*), type checks it with respect to the environment *Env*, and maps it to an expression (of type *ExprT*) in the simply-typed lambda calculus. It is realized as a function *compile* :: *Env* → *ExprS* → *ExprT*:

-- concrete syntax and its abstract syntax in Haskell

$$\begin{array}{ll}
e_S ::= x \mid e_S\ e_S \mid & \textbf{data}\ \mathit{ExprS} = \mathit{VarS\ Ident} \mid \mathit{AppS\ ExprS\ ExprS} \\
\quad \lambda x^u.e_S & \qquad\qquad\qquad \mid \mathit{LamS\ Ident\ Ident\ ExprS} \\
e_T ::= u \mid e_T\ e_T \mid & \textbf{data}\ \mathit{ExprT} = \mathit{VarT\ Ident} \mid \mathit{AppT\ ExprT\ ExprT} \\
\quad \lambda u.e_T & \qquad\qquad\qquad \mid \mathit{LamT\ Ident\ ExprT} \\
\tau ::= \alpha \mid \mathit{Int} \mid \tau \to \tau & \textbf{data}\ \mathit{Ty} = \mathit{TyVar\ Var} \mid \mathit{TyInt} \mid \mathit{TyArr\ Ty\ Ty}
\end{array}$$

Before delving into the actual implementation, we first give a specification of the type system, together with translation rules.

$$\boxed{\Gamma \vdash e_S : \tau \rightsquigarrow e_T}$$

innermost x^u of all:

$$\frac{x^u : \tau \in \Gamma}{\Gamma \vdash x : \tau \rightsquigarrow u}\ \textsc{var} \qquad \frac{\Gamma \vdash f : \tau_1 \to \tau_2 \rightsquigarrow f' \quad \Gamma \vdash a : \tau_1 \rightsquigarrow a'}{\Gamma \vdash f\ a : \tau_2 \rightsquigarrow f'\ a'}\ \textsc{app} \qquad \frac{\Gamma, x^u : \tau_1 \vdash e : \tau_2 \rightsquigarrow e'}{\Gamma \vdash \lambda x^u.e : \tau_1 \to \tau_2 \rightsquigarrow \lambda u.e'}\ \textsc{lam}$$

Each lambda is assumed to be annotated with a unique identifier u. Rule VAR is rather informal[2]. Of all the bindings for x with the right type τ, the innermost one is to be chosen. Its annotation u is used as name in the translation. The rule APP is standard. In rule LAM, the type of a binding is appended to the environment. The annotation of the binding is used as the name of the binding in the translation.

Given an (empty) environment, a SHADOW expression, and optionally a type, we can manually construct a derivation tree using these translation rules. The lookup of a binding poses a challenge due to context sensitivity. For example, for $\lambda x^{u_1}.\lambda x^{u_2}.f\ x$, the choice between translations $\lambda u_1.\lambda u_2.f\ u_1$ and $\lambda u_1.\lambda u_2.f\ u_2$ depends on what the program, where this expression occurs in, states about the type of f and the type of the entire expression by itself. When

[1] The language SHADOW can be used to model typed disambiguation of duplicately imported identifiers from modules. However, SHADOW is only an example. Its design rationale is out of the scope of this section.

[2] Actually, the specification itself is incomplete and informal. We stress that our goal is not to rigorously discuss and prove properties about SHADOW. Instead, we show RulerCore and its concepts. The translation for SHADOW acts as illustration.

the context imposes insufficient restrictions to find a unique solution, the VAR states that we should default to the innermost possibility[3].

5.2.2 Relation to Attribute Grammars

We focus on writing an implementation for the above translation rules with attribute grammars using RulerCore. For each relation in the above specification, we introduce a nonterminal in the RulerCore code. The parameters of the relations become attributes of these nonterminals, thus also the expression part which is normally implicit in an AG based description. Derivation rules become productions, and their contents we map to semantic rules. The productions do not contain terminals: only the values of attributes determine the structure of the derivation tree. Thus, the grammar defines the language of derivation trees for the translation rules. Note that this differs from the standard AG approach, where a single specific parameter of the relations fully determines the shape of the derivation tree.

Operationally, the algorithm, which specifies the construction of the derivation tree, picks a production, recursively builds the derivations for the children of the production, computes attributes, and if there is a mismatch between the value of an attribute and what is expected of it, backtracks to the next production.

We treat productions a bit differently in order to capture the gradual process of type inference. Final decisions about what productions are chosen to make up the derivation tree cannot be made until sufficient information is available. Therefore, we construct the derivation tree in one or more sequential passes called visits.

As key feature of RulerCore, the grammar may contain productions per visit of a nonterminal. To make the distinction clearer, we call these productions *clauses*. During the construction of the part of a derivation for a visit, we try to apply the available clauses to build the portion of the derivation tree for that visit. When successful, we finalize the choice for that clause (similar to the cut in Prolog). The next visit can thus assume that those parts of the derivation tree constructed in previous visits is final. Moreover, we often wish to repeat a visit when type information was discovered that sheds new light upon decisions taken earlier during the visit. The upcoming example code shows this behavior several times.

In a conventional AG, each node in the AST is associated with exactly one production. In RulerCore, however, we may at each visit *refine* the production that is associated with the node. So, we can regard our approach as having our productions organized as the leafs of a tree of clauses, and at each next visit we specialize the choice by going down one of the paths in the tree. Paths in this tree determine actual productions.

A RulerCore program is a Haskell program augmented with attribute grammar code. We generate plain Haskell code from such a program. For each production of RulerCore we generate a coroutine in Haskell. These coroutines are encoded as continuations. The following is a code snippet of a RulerCore program. We explain the syntax further below:

```
module MyCompiler where
data ExprS = ...    -- Haskell data declaration
```

[3] This example has a strong connection to context reduction in Haskell. The inference rules are type-directed. Such rules may be overlapping, and the choice of which rule to apply in a typing derivation may be ambiguous.

```
itf TransExpr ...     -- RulerCore nonterminal and attribute declaration
itf TransExpr ...     -- additional attribute declarations for nonterminal
ones = 1 : ones       -- Haskell binding
translate = sem transExpr :: TransExpr    -- embedded production
              ... rules ...                -- rules of RulerCore
sem transExpr ... rules ...    -- additional rules for production
sem transExpr ... rules ...    -- even more rules
```

With an itf-block, we define a nonterminal and its attributes. With a sem-block, we we define a production, and its clauses and rules, inside a Haskell expression. The sem-block is substituted with the coroutine that is generated for *transExpr*. This coroutine is thus a Haskell expression that, and is bound to the Haskell identifier *translate*. Additional clauses and rules can be given in separate toplevel sem-blocks.

The coroutines that we generate from productions are known as visit functions [Swierstra and Alcocer, 1998]. Inputs to and outputs of the coroutine represent inherited and synthesized attributes respectively. Clauses are mapped to function alternatives of the coroutine. The internal state of the coroutine represents the derivation tree. This state contains instances of coroutines (their closures) that represent the children. An invocation of such a visit function corresponds to a visit in the attribute grammar description. An invocation of a visit function of a root nonterminal thus corresponds to the partial construction of the root node of the derivation tree. Section 5.3 shows the translation to Haskell.

In RulerCore, a production is the root of a tree of clauses. Thus, we can represent a production of a conventional grammar as a production in RulerCore. Alternative, as we do in the example, we can also introduce only one production per nonterminal, and use clauses to represent productions of a conventional grammar. The distinction is mainly technical: productions can be used when the tree is determined before attribute evaluation (the productions form an algebra), whereas clauses can be used to determine the structure of the tree based on attribute values.

5.2.3 Typing Expressions

Figure 5.1 gives a rudimentary sketch of a derivation tree and some of the nonterminals and productions that we introduce. The root node corresponds to production *wrapper* of nonterminal *Compile*. It has a child which is related to production *transExpr* of nonterminal *TransExpr*. Production *transExpr* consists of clauses *exprVar*, *exprLam*, etc. for the various forms of syntax of SHADOW. In locating an identifier in the environment, three nonterminals play a role. Clause *lookupTy* of nonterminal *Lookup* has a list of children, each with nonterminal *LookupMany*. Each child is associated with a clause *lookupLam* of nonterminal *LookupOne*, and represent a choice for a binding. The dotted line points to that binding. At the end of inference, at most one of these choices remains per child with nonterminal *Lookup*.

We declare a type *TransExpr* for the production *transExpr*, which describes the *interface* of a nonterminal. The interface declares the visits and attributes of a nonterminal:

```
itf TransExpr           -- declaration of nonterminal TransExpr
  inh env :: Env        -- inherited attr (belongs to some visit)
```

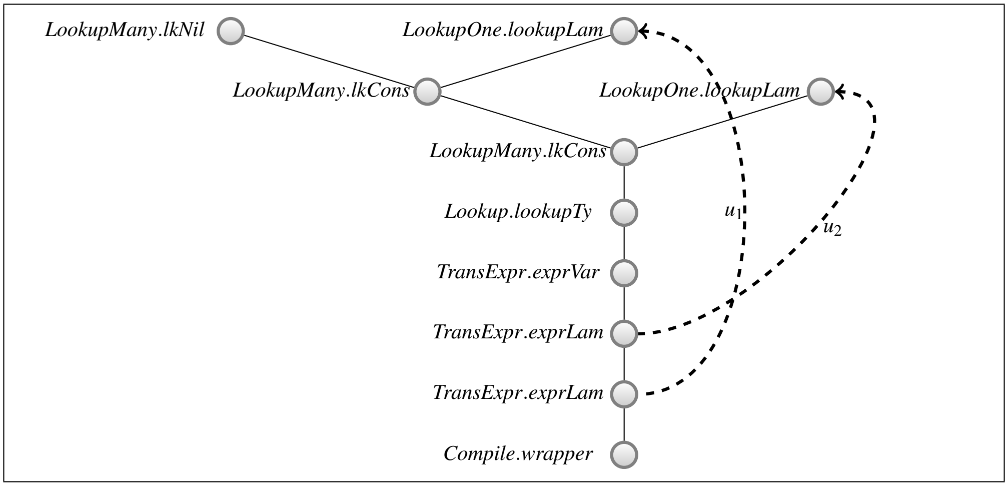


Figure 5.1: Sketch of a derivation tree for $\lambda x^{u_1}.\lambda x^{u_2}.x$

```
    visit dispatch         -- declares a visit
      inh expr :: ExprS    -- inherited attr (belongs to visit dispatch)
  type Env = Map Ident [LookupOne]    -- shown further below
```

In this case, a production of a nonterminal *TransExpr* has a single visit named *dispatch*, and two inherited attributes. The attribute *expr* contains the expression to translate. We define more attributes and visits on *TransExpr* below.

The visits are totally ordered based on value-dependencies between attributes that are derived from the clauses in the whole program. This order is constructable when the attribute dependencies are acyclic. Chapter 3 explains how to derive this order, and Chapter 4 generalizes that approach. Attribute *expr* is declared explicitly for visit *dispatch* (note the indenting). The *env* attribute is not declared for a particular visit. The latest visit it can be allocated to is determined automatically.

From the interface of a nonterminal, a Haskell type for the coroutines is generated. Also, wrapper functions (Section 1.3.1) to invoke the coroutines and access or provide values for attributes from the Haskell code are generated from the interface.

We introduce a production *transExpr* with nonterminal *TransExpr* using a sem-block, embedded in Haskell code. This sem-block is translated to a coroutine in plain Haskell:

```
    -- embedded toplevel Haskell code:
  translate = sem transExpr :: TransExpr
```

We bind it to the Haskell name *translate*, such that we can refer to it from the Haskell code. The nonterminal name *transExpr* is global to the entire program. The Haskell name *translate* follows Haskell's scoping rules.

If we visit a *transExpr*-node for the first time, we have to see what kind of expression we have at hand. We do so by defining a number of alternative ways to deal with the node by introducing a couple of named clauses. For each of the clauses, we subsequently introduce further sem-rules to determine what has to be done. Unlike most of the other code that we give in this section, the order of appearance is relevant for clauses. Operationally, clauses are tried in that order:

```
sem transExpr                    -- production with clauses
  clause exprVar  of dispatch    -- typical name of first visit
  clause exprApp  of dispatch
  clause exprLam of dispatch
```

```
sem exprVar   -- clause with rules (or clause of next visit)
    -- semantic rule (i.e. to match against attributes)
    -- semantic rule (i.e. to define a child)
    -- perhaps clause of next visit (in scope of this clause)
```

Clauses provide a means of scoping. For example, we typically declare clauses of the next visit in the scope of the parent clause (i.e. clause taken at the previous visit). These inherit all the attributes and children in scope of that clause. Otherwise, they only inherit the common attributes and children. This enforces as well that the embedded clause takes place after the enclosing clause.

With a clause, we associate a couple of semantic rules, all of which may fail and cause backtracking, may have an effect on the derivation tree we are constructing, or lead to a runtime error.

- **match** *pattern* = *code*   -- match-rule
  match (*VarS loc.nm*) = *lhs.expr*   -- example

  The Haskell pattern *pattern* must match the value of the right hand side. Evaluation of the rule requires the full pattern match to take place, or causes a backtrack to the next clause.

  Variables in the pattern refer to attributes, and have the form *childname.attrname*. The child name *lhs* is reserved to refer to the attributes of the current node. Furthermore, child name *loc* is a virtual child that conveniently stores local attributes, analogously to local variables. In the example, *lhs.expr* thus refers to the inherited attribute *expr* of the current node.

- *pattern* = *code*   -- assert-rule (not prefixed with a keyword)

  The meaning of an assert-rule is similar to the match-rule, except that the match is expected to succeed. If not, its evaluation aborts with a runtime error.

- **child** *name* :: *I* = *code*   -- child-rule
  child *fun* :: *TransExpr* = *translate*   -- example

  In contrast to a conventional attribute grammar, we construct the tree during attribute evaluation. The rule above creates a child with the given *name*, described by a non-terminal with interface *I*, and defined by the coroutine *code*. For example, *code* could be the expression *translate*, or a more complex expression. Further below, we show an example where the code for a child is provided by an attribute. Evaluation of the child rule creates a fresh instance of this coroutine. This child will thus have its own set of attributes defined by *I*.

- **default** *name* $[=f]$ -- default rule

Provides a default definition for all synthesized attributes named *name* of the production and all inherited attributes of the children that are in scope. This default definition applies only to an attribute if no explicit definition is given. We come back to this rule further below.

We introduce more forms of rules further below.

The evaluation order of rules is determined automatically based on their dependencies on attributes. Rules may refer to attributes defined by previous rules, including rules of clauses of previous visits. Similarly, attributes are mapped automatically to visits based on requirements by rules. Cyclic dependencies are considered to be a static error. The rules may be scheduled to a later visit, except for match-rules. These are scheduled in the visit of the clause they appear in. Visits to children are determined automatically based on dependencies of attributes of the children. If a visit to a child fails, which is the case when none of the children's clauses applies, the complete clause backtracks as well.

The following sem-block defines rules for clause *exprVar*. It states that the value of attribute *lhs.expr* must match the *VarS* constructor:

```
sem exprVar    -- clause exprVar of nonterminal TransExpr
  match (VarS loc.nm) = lhs.expr   -- if succesful, defines loc.nm
```

An attribute grammar distinguishes two categories of attributes: inherited and synthesized. The names of attributes within the same category need to be distinct. Attribute variables in patterns refer to *synthesized* attributes of *lhs*, or *inherited* attributes of the children. Likewise, attribute variables in the right-hand side of a match refer to *inherited* attributes of *lhs* or *synthesized* attributes of the children. This ensures that attribute occurrences are uniquely identifyable.

The following clause for *exprApp* demonstrates the use of child-rules. It introduces two children *f* and *a* with interface *TransExpr*, represented as instances of the *translate* coroutine:

```
sem exprApp    -- clause exprApp of nonterminal TransExpr
  match (AppS loc.f loc.a) = lhs.expr   -- test for AppS
  child f :: TransExpr = translate      -- recurses on f
  child a :: TransExpr = translate      -- recurses on a
  f.expr     = loc.f                    -- passes loc.f as expr-attribute
  a.expr     = loc.a                    -- passes loc.a as expr-attribute
  f.env      = lhs.env                  -- passes the environment topdown
  a.env      = lhs.env                  -- passes the environment topdown
```

The last two lines express that the environment is passed down unchanged. We may omit these rules, and write the rule **default** *env* instead. When a child has an inherited attribute *env*, but no explicit rule has been given, and the production has *lhs.env*, then that value is automatically passed on:

```
sem transExpr    -- for all clauses of transExpr
   default env
```

There are several variants of default-rules. We show further below a default rule for synthesized attributes.

We skip the clause *exprLam* for now, and consider types and type inference first. As usual with type inference, we introduce type variables for yet unknown types, and compute a substitution that binds types to these variables:

```
itf TransExpr              -- extends the interface of TransExpr
   syn ty      :: Ty       -- synthesized attr in unspecified visit
   chn subst₁ :: Subst    -- chained attr in unspecified visit
data Subst    -- left implicit: a mapping from variables to types
```

The chained attribute $subst_1$ stands for both an inherited and synthesized attribute with the same name. We can see this as a substitution that goes in, and comes out again updated with new type information that became available during the visit. We get automatic threading of the attribute through all children that have a chained attribute with this name, using the default-rule:

```
sem transExpr    default subst₁
```

To deal with types and substitutions, we define several helper nonterminals:

```
itf Lookup                -- finds a pair (nm, ty′) ∈ env
   inh nm    :: Ident     -- such that ty′ matches ty.
   inh ty    :: Ty
   inh env   :: Env
itf Unify                 -- computes a substitution s such
   visit dispatch         -- that s (ty₁) equals s (ty₂), if
      inh ty₁ :: Ty        -- possible. Attributes for the
      inh ty₂ :: Ty        -- substitution and errors added below.
itf Fresh                 -- produces a fresh type
   syn ty    :: Ty
   chn subst :: Subst
lookup = sem lookupTy :: Lookup
unify  = sem unifyTy  :: Unify
fresh  = sem freshTy  :: Fresh
```

The implementation of *fresh* delegates to a library function *varFresh* on substitutions:

```
sem freshTy
   (lhs.subst, loc.var) = varFresh lhs.subst
   lhs.ty = TyVar loc.var
```

We did not explicitly declare any visits for nonterminal *Fresh*. Therefore, it consists of a single anonymous visit. When a production does not specify a visit-block, an anonymous visit-block is implicitly defined. Similarly, when a production does not define clauses for a production, an anonymous clause-block is implicitly defined.

We can wrap any Haskell function, including a data constructors, as a production, and represent an application of this function as child of the production (Section 1.3.7). This is convenient in case of *fresh*, because we use default rules to automatically deal with the substitution attribute.

Both *fresh* and *lookup* are of use to refine the implementation of *exprVar*. With *fresh* we get a fresh variable to use as the type of the expression. The Lookup-child then ensures that at some point this fresh type is constrained in the substitution to the type of a binding for the variable:

```
sem exprVar     -- repeated sem-block: extends previous one
   child fr :: Fresh    = fresh
   child lk :: Lookup = lookup
   lk.nm = loc.nm     -- pass loc.nm to lk (loc.nm matched earlier)
   lk.ty   = fr.ty        -- pass the fresh type to lk
   lhs.ty = fr.ty         -- also pass it up
```

We pass the substitution to child *fr*, and pass the resulting substitution upwards to the parent node:

```
sem exprVar
   sem exprVar    rename subst := subst_1 of fr
   fr.subst      = lhs.subst_1    -- pass down
   lhs.subst_1 = fr.subst          -- pass up
```

Recall that $subst_1$ is a chained attribute, hence there is an inherited $lhs.subst_1$, and a synthesized $lhs.subst_1$. These names are not ambiguous: the right hand side of the rule refers to the inherited attribute, the left hand side to the synthesized. With a rename-rule, we rename attributes of children to choose a more convenient name, for example to benefit from default-rules. The two explicit rules may actually be omitted, because of default-rule mentioned earlier.

In the application clause, we use the *Unify* nonterminal to express that various types should match:

```
sem exprApp
   child fr :: Fresh = fresh
   child u  :: Unify = unify
   u.ty_1  = f.ty
```

```
    u.ty₂  = TyArr a.ty fr.ty
    lhs.ty = fr.ty
    rename subst := subst₁ of fr    -- for default-rule
```

Rules for the substitution may be omitted. The default-rule threads it properly through the *fr* and *u* children, which (after renaming) both have a $subst_1$ chained attribute.

5.2.4 Unification

So far, the example can be implemented with most attribute grammar systems that operate on a fixed abstract syntax tree [Dijkstra and Swierstra, 2004, 2006b]. In the above example, the choice of productions solemnly depends on the *expr* inherited attribute. The attribute grammar is directly based on the grammar of expressions. In the remainder of this section, we move beyond such systems. For unification, we allow a selection of clauses based on two inherited attributes: the attributes ty_1 and ty_2 of nonterminal *Unify* defined above.

The idea behind unification is to recursively compare these types. If one is a variable, then the other type is bound to that variable in the substitution:

```
  sem unifyTy
    clause matchEqVars of dispatch   -- the same variables
    clause matchVarL   of dispatch   -- variable on the left
    clause matchVarR   of dispatch   -- variable on the right
    clause matchArr    of dispatch   -- both an arrow
    clause matchBase   of dispatch   -- both the same constant
    clause matchFail   of dispatch   -- failure
```

To implement these clauses, we need additional infrastructure to obtain the free variables of a type, and bind a type in the substitution. We omit here the actual implementation, since the implementation similar to other examples in this section:

```
  itf Ftv inh ty     :: Ty      -- determines free vars of ty
          inh subst  :: Subst   -- after applying the subst
          syn vars   :: [Var]
  itf Bind inh var   :: Var     -- appends to subst:
           inh ty    :: Ty      -- [var := ty]
           chn subst :: Subst
  ftv  = sem ftv  :: Ftv        -- implemented with RulerCore
  bind = sem bind :: Bind       -- wrapper around library fun
```

We define several additional attributes on the *Unify* nonterminal. For the synthesized attributes *success* and *errs*, we give a default definition of the form **default** $attr = f$. This function f gets as its first parameter a list of values of attributes *attr* of the children that have this attribute. If f is not given, we use the Haskell function *last* for f (Section 2.1):

```
  itf Unify
    visit dispatch
```

```
      chn subst_1   :: Subst
      syn success   :: Bool    -- True iff unification succeeds
      syn changes   :: Bool    -- True iff any variables were bound
    visit outcome
      inh subst_2   :: Subst   -- take subst_2 more recent as
      syn errs      :: Errs    -- subst_1 for better error messages
  sem unifyTy
    default success = and      -- and [] = True
    default changes = or       -- or [] = False
    default errors  = concat
    loc.ty_1 = tyExpand lhs.subst_1 lhs.ty_1    -- apply subst one level
    loc.ty_2 = tyExpand lhs.subst_1 lhs.ty_2    -- apply subst one level
```

The inherited types need to be compared with what is known in the substitution to ensure that we do not bind to a variable twice. Hence, we introduce attributes $loc.ty_1$ and $loc.ty_2$, which are computed by applying the substitution to $lhs.ty_1$ and $lhs.ty_2$. Their values are shared among all clauses and are computed only once. We match on these values to select a clause:

```
  sem matchEqVars    -- applies if we get two equal vars
    match True  = same lhs.ty_1 lhs.ty_2 ∨ same loc.ty_1 loc.ty_2
    -- embedded Haskell code:
  same (TyVar v_1) (TyVar v_2) | v_1 ≡ v_2 = True
  same _                               = False
  sem matchVarL    -- a yet unknown type left
    match (TyVar loc.var) = loc.ty_1
    loc.ty                = loc.ty_2
  sem matchVarR    -- a yet unknown type right
    match (TyVar loc.var) = loc.ty_2
    loc.ty                = loc.ty_1
  sem matchVarL matchVarR    -- common part of above
    child fr :: Ftv = ftv      -- determine free fr.vars
    fr.ty = loc.ty             -- of loc.ty
    child b :: Bind = bind     -- add substitution
    b.var = loc.var            -- [loc.var := loc.ty]
    b.ty  = loc.ty
    rename subst := subst_1 of fr b
    loc.occurs    = loc.var ∈ fr.vars    -- occur check
    lhs.subst_1   = if loc.occurs then lhs.subst_1 else b.subst_1
    lhs.success   = ¬ loc.occurs
    lhs.changes   = ¬ loc.occurs
    lhs.errs =  if loc.occurs
                then [CyclErr lhs.subst_2 loc.var loc.ty]
```

```
                  else  $[\,]$
    sem $matchArr$     -- $t_1 \to t_2$ left and $t_3 \to t_4$ right
       match $(TyArr\ t_1\ t_2) = loc.ty_1$
       match $(TyArr\ t_3\ t_4) = loc.ty_2$
       child $l :: Unify$      $= unify$   -- recurse with argument types
       child $r :: Unify$      $= unify$   -- recurse with result types
       $l.ty_1 = t_1$ ;   $l.ty_2 = t_3$ ;   $r.ty_1 = t_2$ ;   $r.ty_2 = t_4$
    sem $matchBase$     -- applies when e.g. both are $TyInt$
       match $True = loc.ty_1 \equiv loc.ty_2$
    sem $matchFail$      -- mismatch between types
       $lhs.success$     $= False$
       $lhs.errs$          $= [UnifyErr\ lhs.subst_2\ lhs.ty_1\ lhs.ty_2]$
```

The clauses of *unifyTy* are total, thus there is always one that applies, with *matchFail* as fallback. The visits to unification thus always succeed. Potential problems that arose during unification can be inspected through attributes *success* and *errs*.

We now have the mechanisms ready to deal with the case of a lambda expression. For the type of the binding, we introduce a fresh type *fr.ty*, and add this type together with the name to the environment:

```
    sem $exprLam$
       match $(LamS\ loc.nm\ loc.u\ loc.b) = lhs.expr$
       child $b :: TransExpr = translate$    -- recurse
       child $fr :: Fresh$         $= fresh$
       rename $subst := subst_1$ of $fr$
       $b.expr = loc.b$
       $b.env\ \ = insertWith\ (+\!\!+)\ loc.nm\ [loc.lk]\ env$    -- append
       $lhs.ty\ \ = TyArr\ fr.ty\ b.ty$    -- result type is $fr.ty \to b.ty$
       $loc.lk\ \ =$ sem $lookupLam :: LookupOne$    -- see below
```

Environments are treated in an unconventional way. Instead of transporting the information needed to construct the lookup-derivation tree in *exprVar*, the environment transports a coroutine *loc.lk* defined in *exprLam*. We define the production *lookupLam* locally in *exprLam*, such that the rules of *lookupLam* have access to the attributes of *exprLam*:

```
    itf $LookupOne$     -- nonterminal of *nested* production $lookupLam$
       visit $dispatch$ inh $ty :: Ty$
```

The idea is that we instantiate this coroutine at the *exprVar*, then pass it the expected type of the expression, and determine if the expected type matches the inferred type of the binding. The rules for this nested production (shown further below) have access to the local state (i.e. attributes) of the enclosing production. At the binding-site, we have information such as the type and annotation of the binding, which we need to construct the derivation.

5.2.5 Lookups in the Environment

At *exprVar*, the goal is to prove that there is a binding in the environment with the right type. The overall idea is that we construct all possible derivations of bindings for an identifier, using the *lookupLam* nonterminal mentioned earlier.

When there is only one possibility, we incorporate it in the substitution, and repeat the visits. The extra type information may rule out other derivations, and result in new type information, etc. Eventually a fixpoint is reached. From all the remaining ambiguous derivations, we pick the deepest ones, and *default* to those, by incorporating their changes into the substitution. We then repeat the process from the beginning, until no ambiguities remain. We run this process on the expression as a whole. In more complex examples that have a let-binding, this process could be repeated per binding group. In the purely functional RulerCore language, we encode this necessarily imperative process using repeated invocation of visits combined with a chained substitution.

We show the implementation of the above algorithm in a step by step fashion. Recall Figure 5.1. Three nonterminals play an essential role: *Lookup* is invoked from the *exprVar* clause and delegates to *LookupMany* to create all derivations possible. To create one derivation, *LookupMany* creates *LookupOne* derivations, one for each nested production *lookupLam* that was put for that identifier into the environment at *exprLam*:

```
sem lookupTy    -- invoked from exprVar
  child forest :: LookupMany = lookupMany
  forest.lks    = find [] lhs.nm lhs.env   -- all LookupOnes
  forest.ty     = lhs.ty   -- inherited attr of Lookup
```

The *lks* attribute is a list of coroutines. The coroutine *lookupMany* instantiates each of them, and passes on the *ty* attribute to each:

```
itf LookupMany
  visit dispatch inh lks  :: [LookupOne]
                 inh ty   :: Ty
lookupMany = sem lkMany :: LookupMany
sem lkMany
  clause lkNil of dispatch      -- when lhs.lks is empty
  clause lkCons of dispatch     -- when it has an element
sem lkNil
  match [] = lhs.lks   -- reached end of the list
sem lkCons
  match (loc.hd : loc.tl)  = lhs.lks
  child hd :: LookupOne    = loc.hd              -- taken from list
  child tl :: LookupMany = lookupMany   -- recurse
  tl.lks = loc.tl   -- remainder of the lookups
  default ty   -- passes the types by default downwards to hd and tl
```

If all the matching *lookupsOne*s are reduced to one, we pick that one and return its substitution. Otherwise, we return the substitution belonging to the innermost binding (which has highest *depth*):

```
itf Lookup LookupOne LookupMany
  visit resolve     -- hunt for a derivation
    chn subst :: Subst
    syn status :: Status    -- outcome of the visit
    syn depth :: Int        -- depth of the binding
  visit resolved    -- invoked afterward
data Status = Fails | Succeeds {amb :: Bool, change :: Bool}
isAmbiguous (Succeeds True _) = True
isAmbiguous _                 = False
```

Every visit is invoked at least once, unless it is declared to be hidden for a child. We intend to invoke the *resolve* visit multiple times. We show further below how this is done.

The depth information is easily determined at the binding-site for lambda expressions, with an inherited attribute *depth*, starting with 0 at the top, and incrementing it with each lambda:

```
itf TransExpr   inh depth :: Int
sem transExpr default depth = 0
sem exprLam   b.depth = 1 + lhs.depth
```

The default-rule for an inherited attribute optionally takes a Haskell expression (0 in this case), which is only used when there is no parent attribute with the same name.

The rules for *lookupLam* demonstrate the use of the hide-rule:

```
sem lookupLam     -- defined inside exprLam above
  child m :: Unify = unify         -- try match of binding type
  rename subst_1 := subst of m     -- to use-site type lhs.ty
  hide outcome of m    -- declare not to visit outcome
  m.ty_1 = outer.fr.ty    -- of enclosing exprLam
  m.ty_2 = lhs.ty
  lhs.status = if m.success then Succeeds False m.changes else Fails
  lhs.depth = outer.lhs.depth      -- of enclosing exprLam
```

With *hide*, we state not to invoke a visit and the visits that follow. Referencing to attributes of such a visit is considered a static error.

The rules of the *lkCons* clause represent a choice. If from the attributes of the children can be concluded that one derivation remains, it delivers that one's substitution as result. Otherwise, it indicates that an ambiguous choice remains. The lookup with the highest depth is by construction at the beginning of the list:

```
sem lkNil
  lhs.depth  = 0            -- lowest depth
  lhs.subst  = lhs.subst    -- no change to subst
  lhs.status = Fails
```

```
sem lkCons
  hd.subst = lhs.subst   -- passed down to
  tl.subst  = lhs.subst  -- both
  (loc.pick, lhs.status, lhs.depth, lhs.subst)
     = case hd.status of
         Fails → (False, tl.status, tl.depth, tl.subst)
         Success _ hdc →
           let status′ = case tl.status of
                             Fails          → hd.status
                             Success _ tlc → Success True (hdc ∨ tlc)
           in (True, status′, hd.depth, hd.subst)
```

When a visit is invoked again, we typically want to access some results of a previous invocation. To retain state between multiple invocations of a visits, we allow visits to take visit-local chained attributes. For example, an attribute *decided* for visit *resolve*:

```
sem lookupTy   visit.resolve.decided = False   -- initial value
```

From inside the visit, we can match on these attributes to select a clause. Furthermore, there is an implicit default rule for them:

```
sem lookupTy
  clause lkRunning of resolve        -- no final choice yet,
    match False = visit.decided      -- try again
    visit.decided = isAmbiguous lk.status
    default status depth subst       -- just pass on
  clause lkFinished of resolve   -- made final choice
    match True = visit.decided
    lhs.status    = Success False False   -- no change
    lhs.depth     = 0
    default subst
```

The states of child nodes that are introduced by a previous visit are properly maintained if their visits are also repeated. However, child nodes that are created in a visit are not retained when the visit is repeated. To prevent a created node from being discarded, it is possible to store a node in an attribute. Recall that children are derivations, which are instances of a coroutine, and these are first class values. The detach-rule can exactly be used for this purpose:

```
attr = detach visitname of childname ⟩
```

Evaluation of a detach-rule takes the child *childname* that is evaluated up to but not including visit *visitname*, and stores it in an attribute *attr*.

A detached child can be attached with an attach-rule:

```
attach visitname of childname = expr
```

The Haskell expression *expr* represents a tree in a state prior to visit *visitname*. If *childname* already exists as child, the attach-rule overrules the visits starting from *visitname*.

The *resolve* visits on *lookupTy* are invoked from *resolve* visits of *transExpr*. In map *deflMap*, we maintain the substitutions of ambiguous lookups per depth. These have not been incorporated in $subst_2$ yet. Applying the deepest of those substitutions *defaults* the choice for the corresponding bindings:

```
itf TransExpr
  visit! resolve
    chn subst₂   :: Subst
    syn changes :: Bool    -- True iff subst₂ was affected
    syn deflMap :: IntMap [Subst]   -- defaulting subst/depth
```

The bang at the resolve visit indicates that all attributes must be scheduled explicitly to this visit. No attribute is automatically assigned to this visit. This gives the visit a predictable interface, which is convenient when invoking the visit explicitly, as we do further below:

```
sem transExpr
  default changes = or
  default deflMap = unionsWith (++)
  default subst₂
```

For ambiguous lookups in the *exprVar*, we add to *deflMap*:

```
sem exprVar
  clause varLkAmb of resolve      -- put lk.subst in deflMap
    match (Success True _) = f.status
    lhs.deflMap = singleton lk.depth [lk.subst]
    lhs.subst₂ = lhs.subst₂        -- bypass lk.subst
  clause varLkOther of resolve    -- default rules only
```

To drive the iterations, we introduce a production *iterInner*, which invokes visit *resolve* one or more times. The iterate-rule denotes the repeated invocation of a visit of a child:

```
iterate visitname of childname = expr
```

The expression *expr* represents the coroutine of a special production (*iterNext*, explained further below) that computes the inherited attributes for the visit of the next iteration from synthesized attributes of the previous iteration. The iterations stop when this special production does not have an applicable clause:

```
iterInner = sem iterInner :: ExprTrans
sem iterInner
  child e :: ExprTrans = translate   -- iterInner is an extra node
  e.expr = lhs.expr                  -- on top of the derivation tree
  default ...    -- omitted: same defaults as transExpr
```

```
    iterate $resolve$ of $e = next$    -- until $e.changed$ is $False$
    $lhs.subst_2 = $ let $pairs$  $= toAscList\ e.deflMap + [(0,[e.subst_2])]$
                      $substs = head\ pairs$    -- deepest substitutions
                   in $foldl\ substMerge\ e.subst_2\ substs$    -- apply them
    $lhs.changes = \neg\ (null\ e.deflMap)$
```

This special production has as interface the *contravariant* interface of the visit *resolve* of *ExprTrans*, i.e. the inherited attributes turn to synthesized attributes, and vice versa. The triple instead of dual colons indicate this difference:

```
$next$ = sem $iterNext ::: ExprTrans.resolve$    -- one anonymous clause
  match $True = lhs.changes$    -- stops when there are no changes
  default $subst_2$    -- pass prev $subst_2$ into the next iter
```

Finally, we introduce a production *wrapper*, which forms the root of the derivation tree and invokes the visits on the derivation for expressions, including again an iteration of the inner loop:

```
itf $Compile$ inh $expr$  $:: ExprS$
              inh $env$   $:: Env$
              syn $subst :: Subst$
              syn $ty$    $:: Ty$
$compile$ = sem $wrapper :: Compile$
sem $wrapper$
  child $e :: TransExpr = iterInner$
  default $env\ expr\ ty$
  iterate $resolve$ of $e = next$    -- repeat the inner loop
  $lhs.subst = e.subst_2$
```

5.2.6 Translation to the Target Expression

The code so far computes the information needed to translate the source expression. The shape of the derivation is determined, and after iterations, $subst_2$ contains the substitution for the types. We wrap up with generating the target expression as attribute *trans* and collecting the errors:

```
itf $ExprTrans$  visit $generate$
                   inh $subst_3 :: Subst$
                   syn $trans$  $:: ExprT$
                   syn $errs$   $:: Errs$
sem $exprVar$    $lhs.trans = VarT\ lk.nm'$    -- $lk$ delivers the name
sem $exprApp$    $lhs.trans = AppT\ f.trans\ a.trans$
sem $exprLam$    $lhs.trans = LamT\ loc.u\ b.trans$
sem $exprDeriv$ default $errs = concat$
```

```
itf Compile   syn trans :: ExprT
sem wrapper default trans
                 e.subst₃ = e.subst₂
```

The lookupTy nonterminal delivers the name for a variable. The alternatives were constructed in iterations of the *resolve* visits, and stored in the *loc.mbDeriv* attribute. We take it out and continue from there. From the derivations of nonterminal *lkMany*, we pick the name for the first one that has *loc.pick* equal to *True*:

```
itf Lookup   visit resolved
                syn nm′ :: Ident
                syn errs :: Errs
sem lookupTy
  lhs.errs   = maybe [Err_unresolved lhs.nm] (const [ ]) lk.mbNm
  lhs.nm′    = maybe lhs.nm id lk.mbNm
itf LookupMany
  syn mbNm :: Maybe Ident
sem lkNil    lhs.mbNm = Nothing
sem lkCons lhs.mbNm = if loc.pick then Just hd.nm′ else tl.mbNm
itf LookupOne   syn nm′ :: Ident            -- use u as name
sem lookupLam lhs.nm′ = outer.loc.u   -- defined in exprLam
```

The remaining code of the compiler invokes the coroutine generated from the *compile* production. It provides the *ExprS* expressions, and obtains the type and an *ExprT* back. We omit these details.

5.2.7 Discussion

Performance. Clauses introduce backtracking. In the worst case, this leads to a number of traversals that are exponential in the size of the (longest intermediate) tree. In practice, clause selection is typically a function of some inherited attributes (i.e. deterministic), which only requires a constant number of traversals over the tree. For example, this is the case for RulerCore programs expressible in UUAG. We verified that programs generated from RulerCore exhibit the same time and memory behavior as programs generated from UUAG.

Expressiveness. With attributes, we conveniently compute information in one part of the tree, and transport the information to other parts, which allows context-dependent decisions to be made. The notion of visits gives us sufficient control to steer the inference process.

On the other hand, it is not possible to simply plug a type system in RulerCore and automatically obtain an inference algorithm. We provide the building blocks to write inference algorithms for many type systems, but it is up to the programmer to ensure that the result is sound and complete.

Soundness of a RulerCore program is typically easy to prove. Completeness, however, is a different issue. That largely depends on decisions made about unknown types. With

RulerCore, we make explicit when choices are made, and when visits are repeated. We believe this helps when reasoning about completeness.

Constraint-based inference. We establish the following relation to constraint-based inference: a detached derivation can be seen as a constraint, can be collected in an attribute and solved elsewhere. Solving constraints corresponds invoking visits (such as *resolve* in Section 5.2.5) on the derivation, potentially multiple times.

Solving a constraint may result in more constraints. We store these either in a node's state, or collect them in attributes.

A constraint is typically parametrized with information from the context that created it. We provide access to this context via nested nonterminals, which have access to the attributes of their outer nonterminals.

5.3 Semantics

We define RulerBack, a small core language for Attribute Grammars. We translate a RulerCore program in two steps to Haskell. We first desugar RulerCore. This gives us a RulerBack program. We then translate the latter to Haskell. The separately defined attributes of RulerCore are grouped together in RulerBack, visits are ordered, attributes allocated to visits, covariant interfaces translated to normal interfaces, rules ordered based on their attribute dependencies, and rules augmented with default rules. We omit description of this translation, as it is similar to the frontend of UUAG [Löh et al., 1998]. Instead, we focus on the translation to Haskell, which precisely defines the semantics of RulerBack, and thus forms the underlying semantics of RulerCore.

5.3.1 Syntax

The RulerBack language is Haskell extended with additional syntax for toplevel interface declarations, semantic expressions, and attribute occurrence expressions. The following grammar lists these syntax extensions:

$i ::= \textbf{itf}\ I\ \overline{v}$	-- interface decl, with visits v
$v ::= \textbf{visit}\ x\ \textbf{inh}\ \overline{a_1}\ \textbf{syn}\ \overline{a_2}$	-- visit decl, with atributes a_1 and a_2
$a ::= x :: \tau$	-- attribute decl, with Haskell type τ
$s ::= \textbf{sem}\ x :: I\ t$	-- semantics expr, defines production x
$t ::= \textbf{visit}\ x_1\ \textbf{chn}\ \overline{x_2}\ \overline{r}\ \overline{c}$	-- visit def, with common rules r
$\mid\ \square$	-- end of the visit sequence
$c ::= \textbf{clause}\ x\ \overline{r}\ t$	-- clause definition, with next visit t
$r ::= p \leftarrow e$	-- assert-rule, evaluates monadic e
$\mid\ \textbf{match}\ p \leftarrow e$	-- match-rule, backtracking variant
$\mid\ \textbf{invoke}\ x\ \textbf{of}\ c \leftarrow e$	-- invoke-rule, invokes x on c, while e
$\mid\ \textbf{attach}\ x\ \textbf{of}\ c :: I \leftarrow e$	-- attach-rule, attaches a partially evaluated child
$\mid\ p = \textbf{detach}\ x\ \textbf{of}\ c$	-- detach-rule, stores a child in an attr

$o ::= x.x$ -- expression, attribute occurrence
x, I, p, e -- identifiers, patterns, expressions respectively

There are some differences in comparison with the examples of the previous section. Invocations of visits to children are made explicit through the invoke-rule, which also represents the iterate-rule. Similarly, the attach rule also takes care of introducing children. A visit definition declares number of visit-local chained attributes $\bar{y}$, and has a number of rules to be evaluated prior to the evaluation of clauses. A clause defines the next visit, if any.

The order of appearance of rules determines the evaluation order, which allows them to be monadic. Non-monadic expressions are lifted with *return*. The implementation is parametrizable over any backtracking monad. In this chapter, we use *IO* as example.

5.3.2 Example

The following example is taken from Section 3.3. It demonstrates how to to compute the sum of a list of integers in two visits in RulerBack. In the first visit, the attribute l is inspected to obtain the elements in the list. In the second visit, the elements are summed up:

```
itf S visit v₁ inh l :: [Int] syn ∅          -- decompose list l down
      visit v₂ inh ∅          syn s :: Int   -- compute sum s up
sum′ =   sem sum :: S
   visit v₁ chn ∅ ∅
      clause sumNil₁                          -- when list is empty
         match [] ← return lhs.l              -- match [] = l
         visit v₂ chn ∅ ∅                     -- no visit-local attrs
            clause sumNil₂
               lhs.s ← return 0               -- empty list, zero sum
               □                              -- no next visit
      clause sumCons₁                         -- when list non-empty
         match (loc.x : loc.xs) ← return lhs.l   -- match (x : xs) = l
         attach v₁ of tl :: S ← return sum    -- recursive call
         tl.l ← return loc.xs                 -- l param of call
         invoke v₁ of tl ← noIterations_S     -- visit it to pass l
         visit v₂ chn ∅ ∅
            clause sumCons₂
               invoke v₂ of tl ← noIterations_S   -- visit it to get the sum
               lhs.s ← return (loc.x + tl.x)      -- sum of hd and the tl
               □                                  -- no next visit
```

We translate a RulerBack production to a coroutine, in the form of continuations. From the interface, we generate a type signature for these coroutines:

```
type S       = S_v₁
newtype S_v₁ = S_v₁ ([Int] → IO ((), (S_v₁, S_v₂)))
newtype S_v₂ = S_v₂ (() → IO (Int, (S_v₂, □)))
```

Inherited attributes become parameters, and synthesized attributes are returned as a tuple of results. Each visit also returns two continuations of type S_v_1 and S_v_2 respectively. The first continuation represents the current visit itself (which may be re-invoked with updated internal state), the second continuation represents the next visit, or $\Box$ if there is no subsequent visit. Since no inherited attributes have been declared for the second visit, the continuation of type S_v_2 can actually be represented as a value:

$$\textbf{newtype}\ S\_v_2 = S\_v_2\ (IO\ (Int, (S\_v_2, \Box)))$$

The coroutine *sum′* has *S* as type. Attributes are encoded as a variable *childIattr* or *childOattr*, depending on whether the attribute is an input or output of the clause. Clause selection relies on backtracking in the monad. When a match-statement doesn't match, a failure is generated in the monad, which we *catch* to switch to the next clause.

```
sum′ = S_v1 vis_v1 where                        -- the initial state
  vis_v1 lhs_I l = (                            -- first clause of visit v1
     do [] ← return lhs_I l                     -- match on lhs.l
        let r = S_v1 vis_v1                     -- repetition cont.
            k = S_v2 vis_v2 where               -- next visit cont.
              vis_v2 = (                        -- clause of visit v2
                do lhs_O s ← return 0           -- lhs.s computation
                   let r = S_v2 vis_v2          -- repetition
                       k = □                    -- no next visit
                   return (lhs_O s, (r, k))     -- deliver result v2
                ) ‘catch‘ (\_ → ⊥)              -- no other clause for v2
        return ((), (r, k))                     -- deliver result of visit v1
    ) ‘catch‘ (\_ →                             -- second clause
     do (loc_L x : loc_L xs) ← return lhs_I l   -- match on lhs.l
        tl_O l          ← return loc_L xs       -- inherited attr tl.l
        (S_v1 vis_tl_v1) ← return sum′          -- attach child tl
        ((), (_, S_v2 vis_tl_v2)) ← vis_tl_v1 tl_O l  -- first visit on tl
        let r = S_v1 vis_v1                     -- repetition cont.
            k = S_v2 vis_v2 where               -- next visit cont.
              vis_v2 = (                        -- clause of visit v2
                do (tl_I s, (_, _)) ← vis_tl_v2  -- second visit on tl
                   lhs_O s ← return (loc_L x + tl_I s)  -- lhs.s
                   let r = S_v2 vis_v2          -- repetition
                       k = □                    -- no next visit
                   return (lhs_O s, (r, k))     -- deliver result v2
                ) ‘catch‘ (\_ → ⊥)              -- no other clause for v2
        return ((), (r, k)))                    -- deliver result of visit v1
```

The above code is slightly simplified. Below, we show the general translation.

5.3.3 Translation

We use the following naming conventions from RulerCore names to Haskell names. The right-hand sides of these definitions consist of string concatenations:

$$
\begin{array}{ll}
\mathit{outp}\ \texttt{"loc"}\ x = \texttt{"locL"}\ x & \mathit{inp}\ \texttt{"loc"}\ x = \texttt{"locL"}\ x \\
\mathit{outp}\ \texttt{"lhs"}\ x = \texttt{"lhsS"}\ x & \mathit{inp}\ \texttt{"lhs"}\ x = \texttt{"lhsI"}\ x \\
\mathit{outp}\ c \quad\ \ x = c\ \texttt{"I"}\ x & \mathit{inp}\ c \quad\ \ x = c\ \texttt{"S"}\ x \\
\mathit{outp}\ y \qquad\ \ = \texttt{"visitS"}\ y & \mathit{inp} \qquad\ y = \texttt{"visitI"}\ y \\
\mathit{vis}\ c\ x \qquad = \texttt{"vis\_"}\ c\ \texttt{"\_"}\ x & \mathit{prod}\ x \qquad = \texttt{"sem\_"}\ x \\
\mathit{vis}\ x \qquad\ \ = \texttt{"vis\_"}\ x & \mathit{ity}\ I\ x = I\ \texttt{"\_"}\ x \\
s\ I\ x \qquad i\ I\ x \quad \text{-- respectively, inh and syn attrs of } x \text{ of } I &
\end{array}
$$

Note that an attribute $c.x$ for some child x at an *output position* represents the inherited attribute x of c, and vice versa for attributes at input positions.

The types of the coroutines are generated from an interface declaration:

$$
\begin{array}{ll}
[\![\textbf{itf}\ I\ \overline{v}]\!] \qquad\qquad \rightsquigarrow \textbf{type}\ [\![I]\!] = [\![\mathit{ity}\ I\ x']\!]; \overline{[\![v]\!]_I} & \text{-- } x' \text{ next visit,} \\
[\![\textbf{visit}\ x\ \textbf{inh}\ \overline{a}\ \textbf{syn}\ \overline{b}]\!]_I \rightsquigarrow \textbf{newtype}\ [\![\mathit{ity}\ I\ x]\!] = & \text{-- otherwise ()} \\
\quad [\![\mathit{ity}\ I\ x]\!]\ ([\![\overline{a}]\!] \to \mathit{IO}\ ([\![\overline{b}]\!], ([\![\mathit{ity}\ I\ x]\!], [\![\mathit{ity}\ I\ x']\!]))) &
\end{array}
$$

From these interfaces, we actually also generate wrappers to interface with the coroutines from Haskell code. The translations for them bear a close resemblance to the translation of the attach and invoke rules below.

The clauses of a visit are translated to a function $[\![\mathit{vis}\ x]\!]$ that tries the clauses one by one. This function takes as parameters the coroutines ($[\![\mathit{chlds}]\!]$) of the children in scope prior to invoking the visit, the visit-local attributes $\overline{y}$, and the inherited attributes:

$$
\begin{array}{l}
[\![\textbf{sem}\ x :: I\ t]\!] \qquad \rightsquigarrow \textbf{let}\ [\![\mathit{prod}\ x]\!] = [\![t]\!]_I\ \textbf{in}[\![\mathit{prod}\ x]\!] \\
[\![()]\!]_I \qquad\qquad\quad \rightsquigarrow () \\
[\![\textbf{visit}\ x\ \textbf{chn}\ \overline{y}\ \overline{r}\ \overline{c}]\!]_I \rightsquigarrow \\
\quad \textbf{let}\ [\![\mathit{vis}\ x]\!]\ [\![\mathit{chlds}]\!]\ [\![\mathit{inp}\ \overline{y}]\!]\ [\![\mathit{inp}\ \mathit{lhs}\ (i\ I\ x)]\!] \\
\qquad = \mathit{catch}\ (\textbf{do}\ \{ [\![\overline{r}]\!]; [\![\overline{c}]\!]_{I,x,\overline{y}} \})\ \bot\ \textbf{in}[\![\mathit{ity}\ I\ x]\!]\ [\![\mathit{vis}\ x]\!]
\end{array}
$$

The clauses themselves translate to a sequence of statements, consisting of the translated statements of the semantic rules, and the construction of the two continuations. We partially parametrize both continuations with the updated children:

$$
\begin{array}{l}
[\![[\,]]\!]_{I,v,\overline{y}} \qquad\qquad \rightsquigarrow \mathit{error}\ \texttt{"no clause applies"} \\
[\![\textbf{clause}\ x\ \overline{r}\ t : \mathit{cs}]\!]_{I,v,\overline{y}} \rightsquigarrow \\
\quad \mathit{catch}\ (\textbf{do}\ \{ [\![\overline{r}]\!]; \textbf{let}\ \{ [\![\mathit{inp}\ x\ \overline{y} = \mathit{outp}\ \overline{y}]\!] \} \\
\qquad\qquad ; \textbf{let}\ \{ r = [\![\mathit{ity}\ I\ x]\!]\ [\![\mathit{vis}\ x]\!]\ [\![\mathit{chlds}]\!]\ [\![\mathit{outp}\ x\ \overline{y}]\!] \\
\qquad\qquad\qquad ; k\ = [\![t]\!]_{I,\mathit{chlds}} \} \\
\qquad\qquad ; \mathit{return}\ ([\![\mathit{outp}\ \mathit{lhs}\ (s\ I\ x)]\!], (r, k)) \}) \\
\qquad\quad (\backslash\_ \to [\![\mathit{cs}]\!]_{I,v,\overline{y}})
\end{array}
$$

$$\begin{array}{ll}
[\![()]\!]_{I,ks} & \leadsto () \\
[\![\textbf{visit}\ x\ \textbf{chn}\ \overline{y}\ \overline{r}\ \overline{c}]\!]_{I,ks} & \leadsto [\![\textbf{visit}\ x\ \textbf{chn}\ \overline{y}\ \overline{r}\ \overline{c}]\!]_{I}\ [\![ks]\!]\ [\![outp\ x\ \overline{y}]\!]
\end{array}$$

Semantic rules translate to monadic statements. For the assert-rule, we match using a let-statement, which ensures that a pattern match failure is considered a runtime error, instead of cause backtracking in the monad:

$$\begin{array}{ll}
[\![\textbf{match}\ p \leftarrow e]\!] & \leadsto [\![p]\!] \leftarrow [\![e]\!] \\
[\![p \leftarrow e]\!] & \leadsto x \leftarrow [\![e]\!]; \textbf{let}\ \{\ [\![p]\!] = x\} \qquad \text{-- } x \text{ fresh} \\
[\![\textbf{attach}\ x\ \textbf{of}\ c :: I \leftarrow e]\!] & \leadsto ([\![ity\ I\ x]\!]\ [\![vis\ c\ x]\!]) \leftarrow [\![e]\!] \\
[\![p = \textbf{detach}\ x\ \textbf{of}\ c]\!] & \leadsto \textbf{let}\ \{\ [\![p]\!] = [\![ity\ I\ x]\!]\ [\![vis\ c\ x]\!]\ \}
\end{array}$$

Invoke invokes a visit x (named f in the translation) on child c once, then repeats invoking it, as long as e (named g) succeeds in feeding it new input:

```
⟦invoke x of c ← e⟧ ⇝
  (⟦inp c (s I_c x)⟧, (_, k))
     ← let iter f ⟦outp c (i I_c x)⟧ = do
               { (⟦coIty I_c x⟧ g) ← ⟦e⟧
               ; z@(⟦inp c (s I_c x)⟧, (⟦ity I_c x⟧ f′, _))
                    ← f ⟦outp c (i I_c x)⟧
               ; catch (do { (⟦outp c (i I_c x)⟧, _) ← g ⟦inp c (s I_c x)⟧
                           ; iter f′ ⟦outp c (i I_c x)⟧ })
                       (\_ → return z) }
        in iter ⟦vis c x⟧ (outp c (i I_c x))
  ; let (⟦ity I_c x′⟧ ⟦vis c x′⟧) = k   -- x′ is next visit, or line omitted
```

Finally, we add bangs around patterns to enforce evaluation, and replace attribute occurrences with their Haskell names:

$$\begin{array}{ll}
[\![C\ \overline{p}]\!] & \leadsto\ !(C\ \overline{[\![p]\!]}) \\
[\![(p,..,q)]\!] & \leadsto\ !([\![p]\!],...,[\![q]\!]) \\
[\![c.x]\!] & \leadsto\ ![\![inp\ c\ x]\!] \\
[\![e]\!] & \leadsto e\ [c.x := [\![outp\ c\ x]\!]]
\end{array}$$

The translation exhibits a number of properties. If the RulerCore or RulerBack program is well typed, then so is the generated Haskell program, and vice versa. Furthermore, the translation is not limited to Haskell. A translation similar to above can be given for any language that supports closures.

5.4 Related Work

Attribute grammars as defined by Knuth [1968] are extensions of context free grammars. Typically, an attribute grammar is defined in terms of a context-free abstract grammar of the language to analyze or compile. The attribute evaluator computes attributes of the abstract syntax tree that is determined apriori by a parser. In case of type inference, when the typing relations are not directed by syntax, the derivation tree is not known beforehand. Thus, the derivation tree cannot be expressed directly in terms of attribute grammars, unless higher-order attributes are used [Vogt et al., 1989].

Tree manipulations. There are many extensions to attribute grammars to facilitate changing the tree during attribute evaluation. Silver [Wyk et al., 2008], JastAdd [Ekman and Hedin, 2007] and UUAG [Löh et al., 1998] support higher-order attributes. These grammars allow the tree to be extended with subtrees that are computed from attributes, and subsequently decorated. The responsibility of selecting a production of a higher-order child lies with the parent of that child, and the choice is final. In RulerCore, a child itself selects a clause to make a choice, and a choice can be made per visit.

JastAdd and Aster [Kats et al., 2009], support conditional rewrite rules, which allows rigorous changes to be made to the tree. Coordination between rewriting and attribute evaluation is difficult to express due to mutual influence, especially if the transformations are not confluent. To limit interplay, JastAdd's rewriting of a tree is limited to the first access of that tree.

Many type inference algorithms, especially for type and effect systems, iteratively traverse the tree. Some algorithms construct additional subtrees during this process. Circular Attribute Grammars [Jones, 1990], supported by JastAdd and Aster, iteratively compute circular attributes until a fixpoint is reached. UUAG and Silver can deal with circularity via lazy evaluation with streams. CAGs, however, do not support changes to the tree during these iterations. Stratego's rewrite mechanism that underlies Aster, however, is more general and can change the tree. In RulerCore, a visit may be iterated several times. Each node in the derivation tree can maintain a per-visit state to keep track of newly constructed parts of the tree.

Non-deterministic trees. The attribute grammar systems above have in common that they massage a tree until it has the right form. Alternatively, a tree can be constructed non-deterministically, using e.g. logic programming languages. The grammar produces only the empty string, and the semantic rules disambiguate the choice of productions. Arbab [1986] show how to translate attribute grammars to Prolog. However, this approach does not allow the inspection of partial *LookupOne* derivations of Section 5.2.5, nor the defaulting, to be implemented easily. With RulerCore, we offer alternative constructions of the tree per visit in combination with backtracking. The notion of a visit provides an intuitive alternative for the *cut* operator.

Prolog-like approaches also offer unification mechanisms to deal with non-determinism in attribute computations. In contrast, we require the programmer to either program unifications and substitutions manually, or use a logic monad combined with a unification in the translation of Section 5.3.

Engelfriet and Filé [1989] show the expressiveness of classes of attribute grammars. Unsurprisingly, deterministic AG evaluators have lower computational complexity bounds compared to non-deterministic ones. With RulerCore, we target large compilers (i.e. UHC), that processes large abstract syntax trees, thus we need the control on the exploration of alternatives that visits offer.

Related attribute grammar techniques. Several attribute grammar techniques are important to our work. Kastens [1980] introduces ordered attribute grammars. In OAGs, the evaluation order of attribute computations as well as attribute lifetime can be determined statically, allowing severe optimizations.

Boyland [1996] introduces conditional attribute grammars. In such a grammar, semantic rules may be guarded. Our clauses-per-visit model also provide guarded rules, but in addition also allow children to be conditionally defined.

Saraiva and Swierstra [1999, chap. 3] describe multi-traversal functions in a functional language (or visit functions [Swierstra and Alcocer, 1998]). These visit functions are one-shot continuations, or coroutines without looping. We improved upon this mechanism to support iterative invocation of visits, thus encoding coroutines with loops.

5.5 Conclusion

We presented the extensions that RulerCore, a conservative extension of ordered attribute grammars, provides to describe type inference algorithms. We explained RulerCore with an extensive example in Section 5.2 and described its semantics in Section 5.3.

RulerCore has several distinct features. Firstly, in contrast to most attribute grammar systems, construction of a derivation tree and the evaluation of its attributes is intertwined in RulerCore. This allows us to define a grammar for the language of derivations of some typing relations, instead of being limited to the grammar of expressions or types.

Secondly, we use the notion of explicit visits to capture the gradual, effectful nature of type inference. Each visit corresponds to a state transition of the derivation tree under construction. These visits may be repeated to form fixpoint iterations.

Thirdly, many inference algorithms reason about what part of the derivation is known, or is still pending, e.g. by means of constraints. In RulerCore, derivation trees are first class and can be inspected by visiting them, which facilitates such reasoning in terms of attributed trees.

6 Case Study with GADTs

Generalized Algebraic Data Types are a generalization of Algebraic Data Types with additional type equality constraints. These found their use in many functional programs, including the development of embedded domain specific programming languages and generic programming.

Recently, several authors published novel inference algorithms and corresponding type system specifications. These approaches tend to be more algorithmic than declarative in nature, and tied to a given compiler infrastructure. This results in complex specifications. For a language implementor, adopting such a complex approach is hard due to conflicting infrastructure and language features. Similarly, type inference is difficult to comprehend for a programmer when the specification is complex.

To make the integration of GADTs in languages easier, we thus need a more orthogonal specification. We present an orthogonal specification for GADTs: the language System $F_{\sim}$, consisting of System F augmented with first-class equality proofs. This specification exploits the Church encoding of data types to describe GADT matches in terms of conventional lambda abstractions.

6.1 Introduction

Generalized Algebraic Data Types (GADTs) allow for additional equalities to hold between types of a data constructor. These equalities must hold when constructing a value with this data constructor. When a match against this constructor succeeds in a pattern match, these equalities hold between the types of the constructor, and may be used to coerce the type of an expression to an equivalent type.

This particular feature found many application areas. Many Haskell-related projects make use of GADTs, such as a diversity of generic programming approaches [Jeuring et al., 2008], and transformations on typed abstract syntax for the implementation of domain-specific languages [Baars et al., 2009]. Therefore, we added support for GADTs in UHC [Dijkstra et al., 2009], the Haskell compiler that we develop at Universiteit Utrecht.

Dijkstra [2005] describes the implementation of UHC as a combination of many specifications. Similarly, we wanted to add GADTs to UHC by taking a specification as guidance. However, it was not straightforward to match existing specifications to UHC's infrastructure, for the following reasons:

- GADT pattern matching is defined in terms of case-expressions. In a programming language such as Haskell, there are several other ways to pattern match, such as in let-bindings and list-comprehensions. Furthermore, Haskell has a concept of lazy pattern matching (irrefutable patterns). It is not directly clear if GADT matching is allowed in these situations, and what the side conditions are.

- For many approaches, it turns out that the effectiveness of inference for GADTs depends on what type information can be derived a priori from type annotations given by the programmer (Section 6.6). Specifications for inference of GADTs use a variety of techniques such as shape inference to describe that process. The implementation of such techniques crosscut the type inference implementation of the compiler, thus we are reluctant to incorporate them, unless absolutely necessary. Additionally, UHC already analyzes a program to derive type information from type signatures. Since the above techniques are intertwined with the specifications, it is not directly clear how to compare our infrastructure with the infrastructure used in the aforementioned specifications.

In this chapter, we present a specification for GADT inference that addresses the above points, and has the following distinguishing features:

- We introduce a variant of System F, named System $F_{\sim}$ (System F-Equality), extended with equality assumptions. In contrast to other GADT specifications, System $F_{\sim}$ does not have syntax for data types or case expressions. These are redundant in our specification, because we exploit the folklore Church encoding of data types as conventional lambda expressions. The result is an abstract and concise description without the redundant syntax, while it is on the other hand more general, because it (necessarily) answers how to treat arbitrary pattern matching, including let-bindings and irrefutable patterns.

- Our specification is orthogonal to type inference infrastructure. Like System F, System $F_{\sim}$ is explicitly typed. These explicit types represent the type information that can be derived using analysis on type signatures and conventional type inference. Our specification thus focusses in isolation on how to infer type coercions.

We thus claim that our specification is *lean*: it is concise because it does not require the notion of data types and case expressions, and abstract because it is orthogonal to infrastructure (in particular regarding type annotation analysis). However, our specification only specifies when explicit type information is needed, not when type annotations may be omitted, which depends largely on the effectiveness of the algorithms we abstracted from.

In an earlier version of this chapter [Middelkoop et al., 2008], we presented this work in terms of an extension of System F_A [Sulzmann et al., 2007], using explicit syntax for data types and case expressions. The encoding in System $F_{\sim}$ is significantly less complex.

Roadmap. We introduce GADTs in Section 6.2, and motivate the design choices for our specification in Section 6.3. We formally define System $F_{\sim}$ and its type system in Section 6.4. In Section 6.5, we define the semantics of System $F_{\sim}$ via a translation to a subset of System F_C, which is a System F-like language with explicit type coercions, defined by Sulzmann et al. [2007]. The relation to various other approaches, we discuss in Section 6.6. Finally, in Section 6.7, we briefly mention our experiences with the integration of GADTs in the UHC.

6.2 Introduction to GADTs

We start this section with an introduction to GADTs. GADTs, combined with the Haskell class system, form an essential ingredient for many Haskell libraries. In this section, we picked two simplified examples as illustration.

6.2.1 Typed Abstract Syntax

One popular use of GADTs is related to the embedding of domain-specific languages in such a way that Haskell's type system can be used to type check the domain-specific language.

A typical implementation of an embedded domain specific language consists of some combinators to construct an abstract syntax tree, and some functionality in the host language to manipulate this abstract syntax tree. After analysis and transformation, the abstract syntax tree is translated to some denotation in the host language in order to use it.

For example, assume that we use Haskell as a host language and embed an expression language containing only tuples and numbers, using the following abstract syntax:

```
data Expr
  = Num Int
  | Tup  Expr Expr
```

The following straightforward translation of the expression to a tuple in the host language does not type check, because the inferred types for the case alternatives are not the same:

```
eval e = case e of
             Num i   → i                      -- expected: Int
             Tup p q → (eval p, eval q)       -- expected: (a,b)
```

We bypass this restriction imposed by the Haskell type system with typed abstract syntax and encode a proof that the generated tuples are type correct. For that, we add a type parameter t to the abstract syntax, which represents the type of the expression, and embed in the constructors a proof (with type *Equal* $t\ t'$) that states that this t is equal to the real type t' of this specific expression (an *Int* for a *Num* and some tuple type for a *Tup*):

```
data Expr t
  =         Num (Equal t Int)      Int
  | ∀a b . Tup  (Equal t (a,b)) (Expr a) (Expr b)
```

Baars and Swierstra [2002] give a definition of this *Equal* data type and some operations, including a function *coerce* that converts the type to a proved equivalent type, and some combinators to construct equality proofs:

```
coerce :: Equal a b → a → b
sym    :: Equal a b → Equal b a
refl   :: Equal a a
trans  :: Equal a b → Equal b c → Equal a c
```

$$\begin{array}{ll} \mathit{congr} \;\; :: \mathit{Equal}\ a\ b \to \mathit{Equal}\ (f\ a)\ (f\ b) & \text{-- has a more general signature} \\ \mathit{subsum} :: \mathit{Equal}\ (f\ a)\ (f\ b) \to \mathit{Equal}\ a\ b & \text{-- than written here} \end{array}$$

A non-bottom value with the type *Equal* τ_1 τ_2 is a proof that the types τ_1 and τ_2 are equal. The *coerce* function applied to such a proof is technically the identity function.

As a hypothetical example, consider the proof p_1 out of assumptions a_1 and a_2:

$$\begin{array}{l} a_1 :: \mathit{Equal}\ v_1\ (\mathit{Int}, v_2) \\ a_2 :: \mathit{Equal}\ v_1\ (v_3, v4) \\ \\ p_1 :: \mathit{Equal}\ (\mathit{Int}, v_2)\ (v_3, v4) \\ p_1 = \mathit{trans}\ (\mathit{sym}\ a_1)\ a_2 \\ \\ p_2 :: \mathit{Equal}\ (\mathit{Int}, v4)\ (v_3, v_2) \\ p_2 = \ldots \mathit{congr} \ldots \mathit{subsum} \ldots \end{array}$$

We continue with the running example in this section and modify the *eval* function such that the case alternatives have the same type, namely the *t* in *Expr t*:

$$\begin{array}{l} \mathit{eval} :: \mathit{Expr}\ t \to t \\ \mathit{eval}\ e = \textbf{case}\ e\ \textbf{of} \\ \qquad\qquad \mathit{Num}\ \mathit{ass}\ i \quad\;\; \to \mathit{coerce}\ (\mathit{sym}\ \mathit{ass})\ i \\ \qquad\qquad \mathit{Tup}\ \;\mathit{ass}\ p\ q \to \mathit{coerce}\ (\mathit{sym}\ \mathit{ass})\ (\mathit{eval}\ p, \mathit{eval}\ q) \end{array}$$

The assumptions used by *eval* need to be proved when constructing values of type *Expr t*, which we achieve by using *refl*:

$$\mathit{Tup}\ \mathit{refl}\ (\mathit{Num}\ \mathit{refl}\ 4)\ (\mathit{Num}\ \mathit{refl}\ 2) \qquad :: \mathit{Expr}\ (\mathit{Int}, \mathit{Int})$$

The important observation to make at this point is that the proofs are a *static* property of the program. Hence, the goal is to construct these proofs automatically.

GHC, the mainstream compiler for Haskell, has built-in support for GADTs. It offers two ways of writing GADTs. One can use qualified type notation, which resembles the example above. Instead of an additional field of the type *Equal*, we write an equality constraint:

$$\begin{array}{l} \textbf{data}\ \mathit{Expr}\ t \\ \quad = \qquad\; (t \sim \mathit{Int}) \quad\;\; \Rightarrow \mathit{Num}\ \mathit{Int} \\ \quad \mid\ \forall a\ b.(t \sim (a,b)) \Rightarrow \mathit{Tup}\ (\mathit{Expr}\ a)\ (\mathit{Expr}\ b) \end{array}$$

Aternatively, one can use special notation for GADTs:

$$\begin{array}{l} \textbf{data}\ \mathit{Expr}\ t\ \textbf{where} \\ \quad \mathit{Num} :: \mathit{Int} \to \mathit{Expr}\ \mathit{Int} \\ \quad \mathit{Tup} :: \mathit{Expr}\ a \to \mathit{Expr}\ b \to \mathit{Expr}\ (a,b) \end{array}$$

These two forms of notation are interchangeable. To convert from the above qualified-type notation to GADT notation, turn each equality constraint into a substitution and apply it exhaustively to the type signature of the constructor. The other way around, given a type siganture of the constructor, take the result type as written (e.g. *Expr* (a,b)), and match it against the actual result type (*Expr t*).

Proofs do not have to be written manually. These are automatically constructed by GHC. The *eval* function can simply be written using either function alternatives or a case expression:

```
eval :: Expr t → t
eval (Num i)   = i
eval (Tup p q) = (eval p, eval q)
```

GHC (version 6.12.1) requires the type signature to be given. We discuss in Section 6.3.2 if this type signature could be inferred and if it is desirable to do so.

6.2.2 Generic Programming

Cheney and Hinze [2003] show how GADTs, called Phantom Types in their work, can be used to implement generic functions that work for many data types. The idea is to have a representation of types as a first class value, then use this representation to navigate generically over a particular value:

```
data Rep t where
  RInt  :: Rep Int
  RChar :: Rep Char
  RList :: Rep a → Rep [a]
  RTup  :: Rep a → Rep b → Rep (a,b)
x :: [(Int,Char)]      r :: Rep [(Int,Char)]
x = [(3,'4')]          r = RList (RTup RInt RChar)
```

In this example, x is a value of some type, and r is a value of a representation of that type.

The following function, for example, traverses a value of any type for which we have a representation, and increments all integers with one:

```
replace :: Rep t → t → t
replace RInt       x     = x + 1
replace RChar      c     = c
replace (RList r)  xs    = map (replace r) xs
replace (RTup a b) (p,q) = (replace a p, replace b q)
```

Values of *Rep t* can typically be obtained automatically at *replace*'s call site using Haskell's class system.

Cheney and Hinze [2003] continue from here, and define a data type for the sum of products representation of data types, and use this representation to define combinators to express generic traversals over arbitrary data types for which there is a representation.

6.3 Design Rationale

In this section, we discuss the design decisions of System $F_\sim$. We are liberal concerning syntax in this section, without loss of generality. In Section 6.4.1, we treat System $F_\sim$ formally.

6.3.1 Church Encoding of Data Types

In other GADT specifications and algorithms, GADT matches are described in combination with case-expressions. In many type system descriptions, however, data types and case expressions are not part of the language, because these are implied by using a Church encoding or Mogensen-Scott encoding [Mogensen, 1992] of the data types. In our specification, we take a similar approach and consider a Church encoding of GADTs.

Church encoding of ADTs. As a prelude, we informally sketch the Church encoding of algebraic data types.

The easiest example to start with are Church booleans. The constructors *True* and *False* can be represented as functions that take two continuation-expressions as parameters and return the appropriate one:

$$
\begin{aligned}
&\textbf{type}\ \mathit{Bool}' = \forall\alpha.\alpha \to \alpha \to \alpha\\
&\mathit{mkTrue}, \mathit{mkFalse} :: \mathit{Bool}'\\
&\mathit{mkTrue}\ \ t\, f = t\\
&\mathit{mkFalse}\ t\, f = f
\end{aligned}
$$

The functions *mkTrue* and *mkFalse* can be used instead of the original constructors. To pattern match against such a constructor, we give it the continuations. For example, the following function:

$$
\begin{aligned}
&\mathit{ifThenElse} :: \mathit{Bool} \to a \to a \to a\\
&\mathit{ifThenElse}\ b\, f\, g\\
&\quad = \textbf{case}\ b\ \textbf{of}\\
&\qquad \mathit{True}\ \ \to f\\
&\qquad \mathit{False} \to g
\end{aligned}
$$

can be encoded as:

$$
\begin{aligned}
&\mathit{ifThenElse} :: \mathit{Bool}' \to a \to a \to a\\
&\mathit{ifThenElse}\ b\, f\, g = b\, f\, g
\end{aligned}
$$

In general, the Church encoding of a data constructor is a function that takes (Church encoded) values for each of its fields, and several continuation functions, one for each constructor. The values are passed on to the appropriate continuation function. For example, given the following data type:

$$
\begin{aligned}
&\textbf{data}\ D\\
&\quad = \mathit{C1}\ \mathit{Int}\\
&\quad \mid\ \mathit{C2}\ D
\end{aligned}
$$

We introduce a (recursive) type D' for Church-encoded Ds, and constructor functions *mkC1* and *mkC2*:

$$\begin{array}{l}
\textbf{type}\, D' = \forall r.(Int \to r) \to (D' \to r) \to r \\
mkC1 :: Int \to D' \\
mkC1\; x\, f1\; \_ = f1\; x \\
mkC2 :: D' \to D' \\
mlC2\; r\; \_f2\; = f2\; r
\end{array}$$

In this example, we assume that built-in types are not encoded. As a technical detail, we require a number of built-in types in order to map recursive types to System F [Urzyczyn, 1996].

When we pattern match in a case-expression, we get a value of type D', and parameterize it with functions that deal with each of the cases. For example, given the following case expression:

$$\begin{array}{l}
\textbf{case}\; x\; \textbf{of} \\
\quad C1\; y \to y+1 \\
\quad C2\; \_ \to \bot
\end{array}$$

In the Church encoding, this corresponds to:

$$x\; (\lambda y.y+1)\; (\backslash\_.\bot)$$

Likewise, we can look at encodings of other forms of pattern matching. A let-binding can be encoded by means of a lambda and application:

$$\textbf{let}\; (C1\; y) = x\; \textbf{in}\; y+1$$

The pattern match takes place when y is evaluated. We can encode such a let-expression as a lambda, if we assume that patterns in a lambda match lazily:

$$(\lambda y.y+1)\; ((\lambda (C1\; y).y)\; x)$$

As continuation functions we take $\bot$, except for the continuation corresponding to the data constructor matched on.

$$(\lambda y.y+1)\; ((\lambda d.d\; (\lambda y.y)\; \bot)\; x)$$

If a let-binding involves multiple variables, we use the Church encoding of tuples.

Haskell has irrefutable (or, lazy) patterns. A match against such pattern always succeeds, and is actually only performed when values of variables that are bound by the pattern are needed. In the following example, the pattern match against *C1* is only performed when the value of y is needed:

$$\begin{array}{l}
\textbf{case}\; x\; \textbf{of} \\
\quad \sim(C1\; y) \to y+1
\end{array}$$

We can write the irrefutable pattern with a let-binding.

$$\begin{array}{l} \textbf{case}\ x\ \textbf{of} \\ \quad z \to \textbf{let}\ (C1\ y) = z \\ \qquad\quad \textbf{in}\ y + 1 \end{array}$$

This expression can be encoded again in a similar way as above.

Church encoding of GADTs. The above approach has as advantage that we only need to consider lambda applications in our specification, and from it, the behavior for case-expressions, let-bindings, etc. can be derived.

In the previous section, we showed that the mechanism underlying GADTs is the construction and application of equality proofs. These proofs are constructed and stored as fields in the constructor, such that they can be taken out when the match succeeds. In the Church encoding, a field has become a lambda. We thus introduce two new forms of expressions. An expression $\tau \sim \sigma$ that builds a proof of the equality between τ and σ, and an expression $\lambda(\tau \sim \sigma).e$ that matches against such a proof and allows it to be used for coercions in e.

For example, for a GADT (written using qualified-type notation):

$$\begin{array}{l} \textbf{data}\ \mathit{Rep}\ a \\ \quad = (a \sim \mathit{Int}) \Rightarrow \mathit{RInt} \end{array}$$

The encoding for *RInt* takes an equality proof, and passes it on to the continuation:

$$\begin{array}{l} \textbf{type}\ \mathit{Rep}'\ a = \forall r.((a \sim \mathit{Int}) \to r) \to r \\ \mathit{mkRInt} :: (a \sim \mathit{Int}) \to \mathit{Rep}'\ a \\ \mathit{mkRInt} = \lambda(a \sim \mathit{Int}).\lambda f.f\ (a \sim \mathit{Int}) \end{array}$$

In order to construct a value of $\mathit{Rep}'\ a$ using *mkRInt*, we first need to pass a proof that $a \sim \mathit{Int}$.

Lazy equalities. We choose the equality-lambda to bind lazily. The following example gives an explanation. The pattern occurs in a let-expression and does not define any variables, which essentially means that it will never be evaluated.

$$\begin{array}{l} f :: \mathit{Rep}\ a \to a \to \mathit{Int} \\ f\ x\ y = \textbf{let}\ \mathit{RInt} = x \\ \qquad\quad\ \ \textbf{in}\ 3 \end{array}$$

In the encoding, x is only evaluated if its equality proof is needed when equalities bind lazily:

$$\lambda(x :: \mathit{Rep}\ a).\lambda(y :: a).(\lambda(a \sim \mathit{Int}).3 :: \mathit{Int})\ (x\ (\lambda(a \sim \mathit{Int}).(a \sim \mathit{Int})))$$

However, if we replace the 3 by y, the equality proof is needed to coerce the type a to *Int*. This subsequently leads to evaluation of x.

It is, however, undesirable that the construction of an equality proof influences the evaluation of the program. With our encoding, we have a model to reason about these effects. For example, it is straightforward to verify that the encoding of GADT matches in a case expression, and the encoding of GADT matches in a lambda, do not influence the evaluation of the program.

6.3.2 Type Inference

Above, we gave type signatures to all expressions involving GADTs. Several authors showed that there are situations where coercions can be inferred without an accompanying type annotation. So far, such approaches are not predictable: it is not intuitive when an annotation may be omitted or when it is obligatory. Several examples in this section illustrate the difficulties regarding inference. We therefore decided to allow coercions only when directed to via a type annotation, and describe how this shows up in the specification.

The main problem regarding GADT inference is that without a type annotation, it is not clear whether or not to coerce a type. In the following example, the pattern match on *Num* brings an equality on *Int* in scope, which may be applied to coerce the type of 3:

$$\begin{array}{l} \textbf{data}\ \mathit{Expr}\ t\ \textbf{where} \\ \quad \mathit{Num}\ \ :: \mathit{Int} \to \mathit{Expr}\ \mathit{Int} \\ \quad \mathit{Tuple} :: \mathit{Expr}\ a \to \mathit{Expr}\ b \to \mathit{Expr}\ (a,b) \\ \mathit{eval}\ (\mathit{Num}\ x) = x \end{array}$$

There are several types possible for *eval*:

$$\begin{array}{l} (1)\ \mathit{eval} :: \forall t\ .\ \mathit{Expr}\ t \to t \\ (2)\ \mathit{eval} :: \mathit{Expr}\ \mathit{Int} \to \mathit{Int} \\ (3)\ \mathit{eval} :: \forall t\ .\ \mathit{Expr}\ t \to \mathit{Int} \end{array}$$

The second type seems appropriate in this case. However, the first type is more general than the second, and the first and third are incomparable. We cannot choose between the first and third type, unless we know precisely how *eval* is to be used. Many inference approaches analyze the usages of such a function to make a choice. When usage of such a function changes due to modifications of the programmer, this may affect the choice, and cause type errors in other parts of the program. This leads to an unpredictable type inference. We choose only to apply equalities when directed to via a type annotation. In the above case, no type annotation is present, we do not apply the equality, which results in type (3).

In the above example, the usage of x in the body seems to suggest a relation to type t in *Expr t*. However, the following example demonstrates that this is not a good criterium, because there are also expressions that have a coercible type, aside from variables occurring in the pattern:

$$\begin{array}{l} \mathit{flex}\ \mathit{True}\ (\mathit{Num}\ x)\ \ = x \\ \mathit{flex}\ \mathit{False}\ (\mathit{Num}\ x) = 3 \end{array}$$

Multiple pattern matches pose additional challenges. In the following example, there are two possible equalities on *Int* to choose from:

$$\mathit{choose}\ (\mathit{Num}\ x)\ (\mathit{Num}\ y) = 3$$

There are now many alternative types possible, including:

$$\begin{array}{l} \mathit{choose} :: \forall t\ s\ .\ \mathit{Expr}\ t \to \mathit{Expr}\ s \to t \\ \mathit{choose} :: \forall t\ s\ .\ \mathit{Expr}\ t \to \mathit{Expr}\ s \to s \\ \mathit{choose} :: \forall t\ s\ .\ \mathit{Expr}\ t \to \mathit{Expr}\ s \to \mathit{Int} \end{array}$$

During inference of the body of *choose*, the problem boils down to making a choice between either coercion to a Skolem constant such as s or t, or not coerce types at all. Again, we refrain from applying an equality unless directed to by a type annotation, and thus end up with the last type.

When there are multiple function or case alternatives, there is often no choice. The equalities have to be applied to have branches with equal types.

$$\begin{array}{ll} \mathit{eval}\ (\mathit{Num}\ x) & = x \\ \mathit{eval}\ (\mathit{Tup}\ a\ b) & = (a,b) \end{array}$$

We still require a type annotation. The design rationale is here that adding additional cases to a function without changing the existing cases, should only make a type less specific. In the above case, adding the *Tup* alternative causes the type of *eval* to change to an incomparable type, which again makes type inference hard to predict from the perspective of the programmer. Also, we point out that in the situation that multiple function or case alternatives are written, the overhead of the annotation (expressed for example in terms of lines of code) decreases.

In our specification, we abstract from the type inference algorithm. Moreover, we require type inference to be completed before the inference of coercions. The language System $F_\sim$, based on System F, is explicitly typed. These explicit types represent the types derived from type annotations and those inferred. This makes it unambiguous where to find the locations where a coercion is needed: at those locations where the actual type does not match the expected type (Section 6.4.5). What remains is to specify how coercions are inferred, and that we make explicit in the specification.

6.4 Specification

In this section, we present our specification for GADT inference. We start with the language System $F_\sim$ in Section 6.4.1, present System $F_\sim$'s typing rules in Section 6.4.2, and show how to deal with equality proofs in Section 6.4.3.

6.4.1 System F-Equal

Figure 6.1 lists the syntax of System $F_\sim$, which consists of System F with the addition of equality proofs. These additions consist of:

- The abstraction $\lambda(\tau \sim \sigma).e$ matches against an equality proof for $\tau \sim \sigma$, and brings it in scope by adding it to the environment. The equalities in scope are called equality assumptions, and can be used in equality proofs. The process of constructing a proof is fully automatic.

$$
\begin{array}{lll}
e,f,g,a \in \mathit{Expr} & & \\
\quad ::= x & \text{(E.VAR)} \\
\qquad \mid \tau \sim \sigma & \text{(E.EQ)} \\
\qquad \mid \lambda x.e & \text{(E.LAM.ABS)} \\
\qquad \mid \lambda \alpha.e & \text{(E.UNIV.ABS)} \\
\qquad \mid \lambda(\tau \sim \sigma).e & \text{(E.EQ.ABS)} \\
\qquad \mid f\ e & \text{(E.APP.EXPR)} \\
\qquad \mid f\ \tau & \text{(E.APP.UNIV)} \\
\\
x,y \ \in \mathit{Ident} \\
\alpha,\beta \in \mathit{TyIdent}
\end{array}
\qquad
\begin{array}{ll}
\tau,\rho,\rho,\sigma \in \mathit{Type} & \\
\quad ::= \alpha & \text{(T.VAR)} \\
\qquad \mid \sigma \to \tau & \text{(T.ARR)} \\
\qquad \mid \forall \alpha.\tau & \text{(T.FORALL)} \\
\qquad \mid \tau \sim \sigma & \text{(T.EQS)} \\
\\
\Gamma \in \mathit{Env} & \\
\quad ::= \Gamma, x :: \tau, & \text{(E.TY)} \\
\qquad \mid \ \Gamma, \alpha & \text{(E.VAR)}
\end{array}
$$

Figure 6.1: Syntax of System $F_{\sim}$

- The proof expression $(\tau \sim \sigma)$ constructs an equality $\tau \sim \sigma$.
- The type language contains a type for equality proofs. Furthermore, we assume a number of primitive types to be present in the type language, such as *Int*.

Environments record Skolem constants introduced by type abstraction, and types for identifiers. Since we present equality proofs as conventional types, these can be stored in the environment as well, when we give them a name.

As illustration, we encode the following fragment of a program that generically increments integers in data types:

$$
\begin{array}{l}
\textbf{data}\ \mathit{Rep}\ \alpha\ \textbf{where} \\
\quad \mathit{RInt} :: \mathit{Rep}\ \mathit{Int} \\
\mathit{inc} :: \mathit{Rep}\ \alpha \to \alpha \to \mathit{Int} \\
\mathit{inc} = \lambda r.\lambda x.\textbf{case}\ r\ \textbf{of} \\
\qquad\qquad\qquad \mathit{RInt} \to x + 1 \\
z :: \mathit{Int} \\
z = \mathit{inc}\ \mathit{RInt}\ 3
\end{array}
$$

As an intermediate step, we write:

$$
\begin{array}{l}
\textbf{type}\ \mathit{Rep}'\ \alpha = \forall \beta.((\alpha \sim \mathit{Int}) \to \beta) \to \beta \\
\mathit{mkRInt} :: \alpha \sim \mathit{Int} \to \mathit{Rep}'\ \alpha \\
\mathit{mkRInt} = \lambda \mathit{eq}.\lambda f.f\ \mathit{eq} \\
\mathit{inc} :: \mathit{Rep}'\ \alpha \to \alpha \to \mathit{Int} \\
\mathit{inc} = \lambda r.\lambda x.r\ (\lambda(\alpha \sim \mathit{Int}).x + 1)) \\
z :: \mathit{Int} \\
z = \mathit{inc}\ (\mathit{mkRInt}\ (\mathit{Int} \sim \mathit{Int}))\ 3
\end{array}
$$

Finally, expressed in System $F_{\sim}$:

$$
\begin{array}{l}
\quad \text{-- } \mathit{mkRInt} \\
\Lambda\alpha.\lambda(eq :: \alpha \sim \mathit{Int}) \\
\quad .\lambda(f :: \forall\beta.((\alpha \sim \mathit{Int}) \to \beta) \to \beta) \\
\quad .f\ \alpha\ eq \\
\quad \text{-- } \mathit{inc} \\
\Lambda\alpha.\lambda(r :: \forall\beta.((\alpha \sim \mathit{Int}) \to \beta) \to \beta) \\
\quad .\lambda(x :: \alpha) \\
\quad .r\ \mathit{Int}\ (\lambda(\alpha \sim \mathit{Int}).(+)\ x\ 1) \\
\quad \text{-- } z \\
\mathit{inc}\ \mathit{Int}\ (\mathit{mkRInt}\ \mathit{Int}\ (\mathit{Int} \sim \mathit{Int}))\ 3
\end{array}
$$

Note that the type of x in *inc* is α, and is required to have type *Int*. In System F, this expression is not correctly typed. It has a valid type in System $F_{\sim}$'s type system, which we discuss below.

6.4.2 Type Rules

The typing relation (rules in Figure 6.2) states that in environment Γ, the expression e has type τ. These rules are syntax directed, except for rule COERCE. The idea is that the shape of the expression determines which rules to apply, and we resort to rule COERCE when there is a mismatch between the types. We illustrate this process briefly with inference for the *inc* example of the previous section, then explain the rules in more detail.

$$\boxed{\Gamma \vdash e : \tau}$$

$$\frac{x :: \tau \in \Gamma}{\Gamma \vdash x : \tau}\ \text{VAR} \qquad \frac{\Gamma \vdash e : \sigma \quad \Gamma \Vdash \sigma \sim \tau \quad \mathtt{ftv}\ \tau \subseteq \mathtt{ftv}\ \Gamma}{\Gamma \vdash e : \tau}\ \text{COERCE} \qquad \frac{\Gamma \Vdash \tau \sim \sigma \quad \mathtt{ftv}\ \tau \subseteq \mathtt{ftv}\ \Gamma \quad \mathtt{ftv}\ \sigma \subseteq \mathtt{ftv}\ \Gamma}{\Gamma \vdash \tau \sim \sigma : \tau \sim \sigma}\ \text{EQ}$$

$$\frac{\Gamma \vdash f : \sigma \to \tau \quad \Gamma \vdash e : \sigma}{\Gamma \vdash f\ e : \tau}\ \text{APP.EXPR} \qquad \frac{\Gamma \vdash f : \sigma \to \tau \quad \mathtt{ftv}\ \sigma \subseteq \mathtt{ftv}\ \Gamma}{\Gamma \vdash f\ \sigma : \tau}\ \text{APP.TY} \qquad \frac{\Gamma, x :: \sigma \vdash e : \tau}{\Gamma \vdash \lambda(x :: \sigma).e : \sigma \to \tau}\ \text{LAM.EXPR}$$

$$\frac{\Gamma, \alpha \vdash e : \tau \quad \alpha \notin \mathtt{ftv}\ \Gamma}{\Gamma \vdash \Lambda\alpha.e : \forall\alpha.\tau}\ \text{LAM.TY} \qquad \frac{\Gamma, x :: \rho \sim \sigma \vdash e : \tau \quad x \notin \Gamma}{\Gamma \vdash \lambda(\rho \sim \sigma).e : (\rho \sim \sigma) \to \tau}\ \text{LAM.EQ}$$

Figure 6.2: Expression type rules

Example. To type the *inc* expression (see Figure 6.3), we apply first rule LAM.TY, then rule LAM.EXPR twice. The former administers the Skolem constant α in the environment, the second two the types of r and x. The type of r, we extract again using the rule VAR, then use rule APP.TY to instantiate the universally quantified β of the type of r to the result type *Int*.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma_1 \vdash r : \forall\beta.(((\alpha\sim\beta)\to\beta)\to\beta)}{\Gamma_1 \vdash r\,Int : ((\alpha\sim Int)\to Int)\to Int} \qquad \Gamma_1 \vdash \lambda(\alpha\sim Int).(+)\,x\,1 : \ldots}{\Gamma_1 \vdash r\,Int\,(\lambda(\alpha\sim Int).(+)\,x\,1) : Int}}{\ldots \vdash \lambda x.r\,Int\,(\lambda(\alpha\sim Int).(+)\,x\,1) : \ldots \to Int}}{\ldots \vdash \lambda r.\lambda x.r\,Int\,(\lambda(\alpha\sim Int).(+)\,x\,1) : \ldots \to \ldots \to Int}}{\emptyset \vdash \Lambda\alpha.\lambda r.\lambda x.r\,Int\,(\lambda(\alpha\sim Int).(+)\,x\,1) : \forall\alpha.\ldots \to \ldots \to Int}$$

$$\dfrac{\dfrac{\dfrac{\Gamma_2 \vdash (+) : \ldots \qquad \dfrac{\Gamma_2 \vdash x : \alpha \qquad \Gamma_2 \Vdash \alpha{\sim}Int}{\Gamma_2 \vdash x : Int}}{\Gamma_2 \vdash (+)\,x : Int \to Int} \qquad \Gamma_2 \vdash 1 : Int}{\Gamma_2 \vdash (+)\,x\,1 : Int}}{\Gamma_1 \vdash \lambda(\alpha\sim Int).(+)\,x\,1 : (\alpha\sim Int)\to Int}$$

$$\begin{aligned}
\Gamma_0 &= \emptyset, 1 :: Int, (+) :: Int \to Int \to Int\\
\Gamma_1 &= \Gamma_0, \alpha, r :: \forall\beta.(((\alpha\sim\beta)\to\beta)\to\beta, x :: \alpha\\
\Gamma_2 &= \Gamma_1, e :: \alpha\sim\beta
\end{aligned}$$

Figure 6.3: Typing Derivation of *inc*

To type the subexpression $(\lambda(\alpha\sim Int).(+)\,x\,1)$, we use rule LAM.EQ to introduce the equality into the environment, bound to a fresh name (not important for now). The subexpression $(+)\,x\,1$ must have type *Int*. Assuming that $(+) :: Int \to Int \to Int$ is in the environment, this means that x must have type *Int*. However, the type we get for x by applying the VAR-rule is α. So, we apply the COERCE-rule to convert the type α to *Int*. This requires us to prove that $\Gamma \Vdash \alpha \sim Int$, for which we give the rules later. This is in this case not difficult, because exactly this equality we added just before into the environment.

Type rules overview. Most of the rules are vanilla System F rules.

Via the `ftv` relation, we state that the types chosen in rules COERCE, EQ, and APP.TY are closed (and well-formed). The `ftv` relation is defined as:

$$\begin{aligned}
\mathtt{ftv}\,\alpha &= \{\alpha\}\\
\mathtt{ftv}\,(\tau\to\sigma) &= \mathtt{ftv}\,\tau \cup \mathtt{ftv}\,\sigma\\
\mathtt{ftv}\,(\forall\alpha.\tau) &= \mathtt{ftv}\,\tau - \{\alpha\}\\
\mathtt{ftv}\,(\tau\sim\sigma) &= \mathtt{ftv}\,\tau \cup \mathtt{ftv}\,\sigma
\end{aligned}$$

We overload `ftv` to work on environments as well:

$$\begin{aligned}
\texttt{ftv}\,\emptyset &= \emptyset\\
\texttt{ftv}\,(\Gamma,\alpha) &= \texttt{ftv}\,\Gamma \cup \{\alpha\}\\
\texttt{ftv}\,(\Gamma, x :: \tau) &= \texttt{ftv}\,\Gamma \cup \texttt{ftv}\,\tau
\end{aligned}$$

Rules COERCE and EQ specify the construction of equality proofs. In the former, the equality proof is used to coerce a type, in the latter to pass it on as a first-class value.

In rules LAM.EXPR, LAM.TY, and LAM.EQ, we extend the environment. Extension of an environment with a binding shadows a possible previous binding with the same name. In rule LAM.EQ, we introduce an equality in the environment, using a fresh name x. This name is of consequence for the next section, but has no meaning in this section.

$$\boxed{\Gamma \Vdash \tau_1 \sim \tau_2}$$

$$\Gamma \Vdash \tau \sim \tau \;\textsc{refl} \qquad \frac{\Gamma \Vdash \sigma \sim \tau}{\Gamma \Vdash \tau \sim \sigma}\;\textsc{sym} \qquad \frac{\begin{array}{c}\Gamma \Vdash \tau \sim \rho\\ \Gamma \Vdash \rho \sim \sigma\end{array}}{\Gamma \Vdash \tau \sim \sigma}\;\textsc{trans} \qquad \frac{x :: \tau \sim \sigma \in \Gamma}{\Gamma \Vdash \tau \sim \sigma}\;\textsc{assum}$$

$$\frac{\begin{array}{c}\Gamma \Vdash \sigma \sim \tau\\ \Gamma \Vdash \rho \sim \rho\end{array}}{\Gamma \Vdash \tau \to \rho \sim \sigma \to \rho}\;\textsc{con.arr} \qquad \frac{\begin{array}{c}\Gamma,\beta \Vdash \tau\,[\alpha := \beta] \sim \sigma\\ \beta \notin \texttt{ftv}\,\tau \cup \texttt{ftv}\,\Gamma\end{array}}{\Gamma \Vdash \forall\alpha.\tau \sim \sigma}\;\textsc{con.univ.left}$$

$$\frac{\begin{array}{c}\Gamma \Vdash \tau \sim \sigma\,[\beta := \alpha]\\ \beta \in \texttt{ftv}\,\Gamma\\ \alpha \notin \texttt{ftv}\,\sigma\end{array}}{\Gamma \Vdash \tau \sim \forall\alpha.\sigma}\;\textsc{con.univ.right} \qquad \frac{\begin{array}{c}x :: (\tau_3 \sim (\forall\alpha.\tau_4)) \in \Gamma\\ y :: (\tau_5 \sim (\forall\alpha.\tau_6)) \in \Gamma\\ \Gamma \Vdash \tau_3 \sim \tau_5\\ x' :: \tau_4 \sim \tau_6 \Vdash \tau_1 \sim \tau_2\\ x' \notin \Gamma\end{array}}{\Gamma \Vdash \tau_1 \sim \tau_2}\;\textsc{sub.univ}$$

$$\frac{\begin{array}{c}x :: (\tau_3 \sim (\tau_4 \to \tau_5)) \in \Gamma\\ y :: (\tau_6 \sim (\tau_7 \to \tau_8)) \in \Gamma\\ \Gamma \Vdash \tau_3 \sim \tau_6\\ x' :: \tau_4 \sim \tau_7, y' :: \tau_5 \sim \tau_8 \Vdash \tau_1 \sim \tau_2\\ x', y' \notin \Gamma\end{array}}{\Gamma \Vdash \tau_1 \sim \tau_2}\;\textsc{sub.arr}$$

Figure 6.4: Equality proof inference rules

6.4.3 Type Conversions

Figure 6.4 lists the inference rules for equality proofs. The rules REFL, SYM and TRANS correspond to the conventional rules of an equality theory. An equality assumption can be applied through rule ASSUM.

In the example of the beginning of this section, we need to prove $\Gamma \Vdash Int \sim Int$ (by means of the REFL-rule), and $\Gamma \Vdash a \sim Int$ (with the ASSUM-rule). In Section 6.2.1 we showed some examples using more involved equality proofs. Simply put, proving an equality is a matter of exhaustively applying all the rules.

To be able to apply coercions deeper into types, we have congruence and subsumption rules for each member of the type language that has a substructure. In our case, for arrows and universal quantification. We do not need nor allow coercions on equality proofs.

With congruence rules, if two types share a common structure, we only need to prove the equality between the components that differ. The other way around, with subsumption rules, we lift a proof on smaller type into a proof on bigger types.

In other specifications Schrijvers et al. [2009], Sulzmann et al. [2007], the subsumption rules e.g. on arrows are typically specified as:

$$\frac{\Gamma \Vdash \tau \to \rho \sim \sigma \to \rho}{\Gamma \Vdash \tau \sim \sigma}\ \textsc{sub.l} \qquad\qquad \frac{\Gamma \Vdash \tau \to \rho \sim \sigma \to \rho}{\Gamma \Vdash \tau \sim \sigma}\ \textsc{sub.r}$$

These rules appear simpler than the rule in our specification. However, we have objections against these rules:

- For any environment Γ, the other rules have a finite number of unique instantiations. However, when we include the above subsumption rule, arbitrary big types can be constructed. Thus the search space can be infinite. In practice, we only need to consider a finite portion of the search space to prove or disprove the equality of two types, but this rule does not force us to.

- The subsumption rule does not give us an intuition when to actually apply this rule. To construct an equality proof, one first decomposes the equality to prove using the congruence rules, then apply subsumption rules to work up to assumptions in the environment. We thus restrict the subsumption rule to assumptions in the environment.

  Our rule simply expresses that if there are two equalities in the environment, and these equalities can be shown equal on one side, then we may assume that each pairwise subtype on the other side is equal as well.

6.4.4 Multiple Derivations

There may be more than one possible way to complete an inference. If there are multiple derivations possible, the equality relation has the nice property that any of them satisfies. Also, if one can be completed, then all of them can be completed.

6.4.5 Algorithm

The specification of this section has a straightforward implementation, even combined with type inference. We first infer types of a program in the conventional way, but for each type conflict, we generate a coercion constraint. At the end of type inference for e.g. a binding group, we try to solve these coercion constraints, using an exhaustive application of the above equality rules. For those constraints that cannot be solved we generate a type error. For those we can, we generate a coercion. In the next section, we show how to generate these coercions.

6.5 Semantics

We define the semantics of System $F_{\sim}$ in terms of System F_C Sulzmann et al. [2007]. We introduction to System F_C, then show the important parts of the translation.

6.5.1 System F-Coercion

System F_C (System F-Coercion) extends System F with equality coercions. It has a built-in syntax to represent values of the *Equal* data type mentioned in the Section 6.2.1.

$$
\begin{array}{lll}
\gamma \in \mathit{Coercion} & & \\
\quad ::= x & & \text{(C.VAR)} \\
\quad \mid \gamma_1\ \gamma_2 & & \text{(C.APP)} \\
\quad \mid \tau & & \text{(C.REFL)} \\
\quad \mid \mathtt{sym}\ \gamma & & \text{(C.SYM)} \\
\quad \mid \gamma_1 \circ \gamma_2 & & \text{(C.TRANS)} \\
\quad \mid \mathtt{right}\ \gamma & & \text{(C.RIGHT)} \\
\quad \mid \mathtt{left}\ \gamma & & \text{(C.LEFT)} \\
\quad \mid \forall \alpha\ .\ \gamma & & \text{(C.UNIV)} \\
\quad \mid \gamma @ \rho & & \text{(C.INST)}
\end{array}
\qquad
\begin{array}{lll}
\hat{e} \in \mathit{Expr} & & \\
\quad \mid e \rhd \gamma & & \text{(E.APP.COE)} \\
\quad \mid \mathbf{case}\ \hat{e}\ \mathbf{of}\ \overline{p \to \hat{e}} & & \text{(E.CASE)} \\
 & & \\
p \in \mathit{Pat} & & \\
\quad \mid C\ \overline{\gamma : \tau \sim \sigma}\ \overline{\alpha : \star}\ \overline{x : \rho} & & \text{(P.CON)} \\
 & & \\
d \in \mathit{Decl} & & \\
\quad \mid \mathbf{data}\ D\ \overline{a}\ \overline{\mid C\ \overline{\tau \sim \sigma}\ \overline{\alpha : \star}\ \overline{\rho}} & &
\end{array}
$$

Figure 6.5: Subset of syntax of System F_C

We limit our explanation to a simplified fragment of System F_C, which contains what we need: System F (lambda abstraction, etc.), data types, case expressions, and coercions. Figure 6.5 lists the extensions to System F. A data constructor C consists of three components: equality coercions, existentials, and fields, respectively, hence it is a representation of the qualified-type style of GADT constructors.

A coercion γ can be applied to an (System F_C) expression $\hat{e}$, denoted as $\hat{e} \rhd \gamma$, which changes the type of $\hat{e}$ according to coercion. A coercion γ of type $\tau_1 \sim \tau_2$ represents a proof of the equality of τ_1 and τ_2. Figure 6.6 lists the typing rules of coercion terms. See Sulzmann et al. [2007] for the full listing.

$$\boxed{\Gamma \vdash \gamma : \tau_1 \sim \tau_2}$$

$$\frac{x :: \tau_1 \sim \tau_2 \in \Gamma}{\Gamma \vdash x : \tau_1 \sim \tau_2}\ \text{VAR} \qquad \frac{\Gamma \vdash \gamma_1 : \tau_1 \sim \tau_3 \qquad \Gamma \vdash \gamma_2 : \tau_2 \sim \tau_4}{\Gamma \vdash \gamma_1\ \gamma_2 : \tau_1\ \tau_2 \sim \tau_3\ \tau_4}\ \text{APP} \qquad \Gamma \vdash \tau : \tau \sim \tau\ \text{REFL}$$

$$\frac{\Gamma \vdash \gamma : \tau_2 \sim \tau_1}{\Gamma \vdash \mathtt{sym}\ \gamma : \tau_1 \sim \tau_2}\ \text{SYM} \qquad \frac{\Gamma \vdash \gamma_1 : \tau_1 \sim \tau_2 \qquad \Gamma \vdash \gamma_2 : \tau_2 \sim \tau_3}{\Gamma \vdash \gamma_1 \circ \gamma_2 : \tau_1 \sim \tau_3}\ \text{TRANS}$$

$$\frac{\Gamma \vdash \gamma : \tau_1\ \tau_2 \sim \tau_3\ \tau_4}{\Gamma \vdash \mathtt{left}\ \gamma : \tau_1 \sim \tau_3}\ \text{LEFT} \qquad \frac{\Gamma \vdash \gamma : \tau_1\ \tau_2 \sim \tau_3\ \tau_4}{\Gamma \vdash \mathtt{right}\ \gamma : \tau_2 \sim \tau_4}\ \text{RIGHT}$$

$$\frac{\Gamma \vdash \gamma : \tau_1 \sim \tau_2 \qquad \alpha \notin \mathtt{ftv}\,\Gamma}{\Gamma \vdash \forall \alpha\,.\,\gamma : \forall \alpha\,.\,\tau_1 \sim \forall \alpha\,.\,\tau_2}\ \text{FORALL} \qquad \frac{\Gamma \vdash \gamma : \forall \alpha\,.\,\tau_1 \sim \forall \beta\,.\,\tau_2}{\Gamma \vdash \gamma @ \rho : \tau_1\,[\alpha := \rho] \sim \tau_2\,[\beta := \rho]}\ \text{INST}$$

Figure 6.6: Coercion type rules

Coercion application $\gamma_1\ \gamma_2$ builds a coercion that applies γ_1 to the function part f and γ_2 to the argument part a of a type application $f\ a$. With transitivity $\gamma_2 \circ \gamma_1$, the second coercion is applied to the result of applying the first coercion. The `right`-coercion extracts the coercion of the arguments from a coercion on a type application $f\ a$.

6.5.2 Translation Overview

We use a type-directed translation. The typing relations have the form:

$\Gamma \vdash e : \tau \leadsto \hat{e}$	-- System F_C-expr $\hat{e}$ is the translation of System $F_\sim$-expr e.
$\Gamma \Vdash \tau_1 \sim \tau_2 \leadsto \gamma$	-- γ is a System F_C-coercion term of type $\tau_1 \sim \tau_2$.

The translation consists of two challenges: we need to represent System $F_\sim$'s equality proofs in System F_C, and need to associate with each equality-proof rule of the previous section a well-typed System F_C-coercion term.

6.5.3 System F-Equality Expressions

The first-class equality proofs in System $F_\sim$ have no direct counterpart in System F_C. However, in System F_C, equality proofs may be stored in a data constructor. For example, for an equality proof of type $\alpha \sim \beta \rightarrow Int$, we can associate a data type:

data $D\ \alpha\ \beta = C\ (\alpha \sim \beta \rightarrow Int)$

In general, we associate a data type $D_{key\,(\tau,\sigma)}\overline{\alpha}$ with a System $F_\sim$ proof of type $\tau \sim \sigma$ ($\overline{\alpha} = \mathtt{ftv}\ \tau \cup \mathtt{ftv}\ \sigma$), as well as a constructor $C_{key\,(\tau,\sigma)}$ storing the System F_C-coercion. We denote

$$\boxed{\Gamma \vdash e : \tau \rightsquigarrow \hat{e}}$$

$$\frac{x :: \tau \in \Gamma}{\Gamma \vdash x : \tau \rightsquigarrow x}\ \textsc{Var} \qquad \frac{\Gamma \vdash e : \sigma \rightsquigarrow \hat{e} \quad \Gamma \Vdash \sigma \sim \tau \rightsquigarrow \gamma \quad \mathtt{ftv}\,\tau \subseteq \mathtt{ftv}\,\Gamma}{\Gamma \vdash e : \tau \rightsquigarrow \hat{e} \triangleright \gamma}\ \textsc{Coerce} \qquad \frac{\Gamma \Vdash \tau \sim \sigma \rightsquigarrow \gamma \quad \mathtt{ftv}\,\tau \subseteq \mathtt{ftv}\,\Gamma \quad \mathtt{ftv}\,\sigma \subseteq \mathtt{ftv}\,\Gamma}{\Gamma \vdash \tau \sim \sigma : \tau \sim \sigma \rightsquigarrow [\![\tau \sim \sigma]\!]\,\gamma}\ \textsc{Eq}$$

$$\frac{\Gamma \vdash f : \sigma \to \tau \rightsquigarrow \hat{f} \quad \Gamma \vdash e : \sigma \rightsquigarrow \hat{e}}{\Gamma \vdash f\ e : \tau \rightsquigarrow \hat{f}\ \hat{e}}\ \textsc{App.Expr} \qquad \frac{\Gamma \vdash f : \sigma \to \tau \rightsquigarrow \hat{f} \quad \mathtt{ftv}\,\sigma \subseteq \mathtt{ftv}\,\Gamma}{\Gamma \vdash f\ \sigma : \tau \rightsquigarrow \hat{f}\ [\![\sigma]\!]}\ \textsc{App.Ty}$$

$$\frac{\Gamma, x :: \sigma \vdash e : \tau \rightsquigarrow \hat{e}}{\Gamma \vdash \lambda(x :: \sigma).e : \sigma \to \tau \rightsquigarrow \lambda(x :: [\![\sigma]\!]).\hat{e}}\ \textsc{Lam.Expr} \qquad \frac{\Gamma, \alpha \vdash e : \tau \rightsquigarrow \hat{e} \quad \alpha \notin \mathtt{ftv}\,\Gamma}{\Gamma \vdash \Lambda\alpha.e : \forall\alpha.\tau \rightsquigarrow \Lambda\alpha.\hat{e}}\ \textsc{Lam.Ty}$$

$$\frac{\Gamma, x :: \rho \sim \sigma \vdash e : \tau \rightsquigarrow \hat{e} \qquad x, y \notin \Gamma}{\Gamma \vdash \lambda(\rho \sim \sigma).e : (\rho \sim \sigma) \to \tau \rightsquigarrow \lambda y.\mathbf{case}\ y\ \mathbf{of} [\![\rho \sim \sigma]\!]\ (x : [\![\rho]\!] \sim [\![\sigma]\!]) \to \hat{e}}\ \textsc{Lam.Eq}$$

Figure 6.7: System $F_{\sim}$-expr translation rules

$$\boxed{\Gamma \Vdash \tau_1 \sim \tau_2 \rightsquigarrow \gamma}$$

$$\Gamma \Vdash \tau \sim \tau \rightsquigarrow [\![\tau]\!]\ \textsc{Refl} \qquad \frac{\Gamma \Vdash \tau^r \sim \tau^l \rightsquigarrow \gamma}{\Gamma \Vdash \tau^l \sim \tau^r \rightsquigarrow \mathtt{sym}\,\gamma}\ \textsc{Sym} \qquad \frac{\Gamma \Vdash \tau_1 \sim \tau_2 \rightsquigarrow \gamma_1 \quad \Gamma \Vdash \tau_2 \sim \tau_3 \rightsquigarrow \gamma_2}{\Gamma \Vdash \tau_1 \sim \tau_3 \rightsquigarrow \gamma_2 \circ \gamma_1}\ \textsc{Trans}$$

$$\frac{x : \tau_1 \sim \tau_2 \in \Gamma}{\Gamma \Vdash \tau_1 \sim \tau \rightsquigarrow x}\ \textsc{Assum} \qquad \frac{\Gamma \Vdash \sigma \sim \tau \rightsquigarrow \gamma_1 \quad \Gamma \Vdash \rho \sim \rho \rightsquigarrow \gamma_2}{\Gamma \Vdash \tau \to \rho \sim \sigma \to \rho \rightsquigarrow \gamma_1 \to \gamma_2}\ \textsc{Con.Arr}$$

$$\frac{\begin{array}{c} \gamma_1 :: (\tau_3 \sim (\tau_4 \to \tau_5)) \in \Gamma \qquad \gamma_2 :: (\tau_6 \sim (\tau_7 \to \tau_8)) \in \Gamma \\ \Gamma \Vdash \tau_3 \sim (\tau_6 \rightsquigarrow \gamma_3) \\ \gamma_5 = \mathtt{right}\ (\mathtt{sym}\,\gamma_1 \circ \gamma_3 \circ \gamma_2) \qquad \gamma_6 = \mathtt{right}\ (\mathtt{right}\ (\mathtt{sym}\,\gamma_1 \circ \gamma_3 \circ \gamma_2)) \\ \gamma_5 :: \tau_4 \sim \tau_7, \gamma_6 :: \tau_5 \sim \tau_8 \Vdash \tau_1 \sim \tau_2 \rightsquigarrow \gamma_4 \\ \gamma_1, \gamma_2 \notin \Gamma \end{array}}{\Gamma \Vdash \tau_1 \sim \tau_2 \rightsquigarrow \gamma_4}\ \textsc{Sub.Arr}$$

Figure 6.8: Equality proof translation rules

with $[\![\tau \sim \sigma]\!]$ the conversion from a System $F_\sim$-proof type to a System F_C data type, or data constructor according to the above procedure. Similarly, $[\![\sigma]\!]$ denotes the translation of a System $F_\sim$ type to a System F_C type, by recursively mapping each coercion type $\tau \sim \sigma$ to $D_{key\,(\tau,\sigma)}\overline{\alpha}$.

Figure 6.7 shows the translation rules, which are the type rules of the previous section extended with the resulting $\hat{e}$ expression.

6.5.4 Translation of Proofs

Figure 6.8 shows how to produce coercion terms from the equality proof. We omitted the rules for universal quantification: these are analogous. The rules REFL, SYM, TRANS, and ASSUM have a direct mapping to a coercion-term.

Each individual rule is a solution to a small puzzle. For a proof of $\tau \sim \sigma$, we combine coercions γ until they have exactly the same type according to coercion type rules in Figure 6.6. For example, we can verify that rule SUB.ARR constructs a coercion with the right type as follows:

$$\begin{array}{ll}
\gamma_1 & :: \tau_3 \sim (\tau_4 \rightarrow \tau_5) \\
\gamma_2 & :: \tau_6 \sim (\tau_7 \rightarrow \tau_8) \\
\gamma_3 & :: \tau_3 \sim \tau_6 \\
\texttt{sym}\,\gamma_1 \circ \gamma_3 \circ \gamma_2 & :: (\tau_4 \rightarrow \tau_5) \sim (\tau_7 \rightarrow \tau_8) \\
\texttt{right}\,(\texttt{sym}\,\gamma_1 \circ \gamma_3 \circ \gamma_2) & :: \tau_4 \sim \tau_7 \\
\texttt{right}\,(\texttt{right}\,(\texttt{sym}\,\gamma_1 \circ \gamma_3 \circ \gamma_2)) & :: \tau_5 \sim \tau_8
\end{array}$$

6.5.5 Correctness

It is easy to show that a well-typed equality-proof is translated to a well-typed coercion term (see above). It is also easy to show that a System $F_\sim$ expression is translated to a well-typed System F_C expression. Because System F_C has type soundness, we thus obtain that System $F_\sim$ also has type soundness.

6.6 Related Work

Several approaches propose limitations to make inference for GADTs tractable. Many specifications and algorithms for GADTs consist of two components: a type information propagation component and a type conversion component. A type information propagation strategy determines what is known type information based on user-supplied type signatures, and what is inferred type information. A type conversion strategy deals with the construction of coercion terms.

6.6.1 Restricting Expressiveness for Tractable Inference

Sulzmann et al. [2006a] show that inference for GADTs is undecidable without a helping hand from the programmer in the form of type annotations, and show that principal typing is lost. They present an inference algorithm that generates implication constraints, and apply a technique called Herbrand abduction [Maher, 2005] to solve these constraints. A solution to these constraints is a normalized constraint that implies the generated constraints. The authors define a notion of a sensible solution; under these conditions the resulting type is principal. These conditions are specified in terms of the algorithm.

We are reluctant to adopt such an approach, because it requires programmers to think in terms of an algorithm. To discover why their program is (not) accepted by the type inferencer, the steps of the algorithm have to be performed by hand.

Schrijvers et al. [2009] recently presented two type systems for GADTs, and the inference algorithm OutsideIn. The first type system is relatively simple, but inference for it is undecidable. The latter is more restricted, but has OutsideIn as a sound and complete inference algorithm. Both specification and inference algorithm are given in terms of implication constraints.

When type inference is combined with coercion inference, additionally a coercion $\tau_1 \sim \tau_2$ may be constructed when τ_1 unifies with τ_2 (the result then corresponds to the REFL rule). The main idea of OutsideIn is that binding of variables during such a unification is not allowed on variables that are free in the environment (of the enclosing let-expression or case-branch). These are called the “untouchable” unification variables. Type checking proceeds as normal, except that a unification of a Skolem constant or with an untouchable requires a coercion to be constructed. The actual construction of these coercions is done after type checking is complete.

The approach is presented as a general solution to inference for GADTs. Although this approach is sound and complete with respect to their specification, their specification still requires type annotations for most functions with GADT patterns. For example, with OutsideIn, a type signature is needed for the following code that uses GADTs in the typical way:

```
data D a where
   C1 :: Int → D Int
   C2 :: Bool → D Bool
   -- needs: f :: D α → α
f x = case x of
   C1 y → y
   C2 z → z
```

The reason is that α is free in the environment, and thus not unified.

This approach still lacks a good specification: it introduces “suspended judgments” to encode a two-phase typing of an expression, which is actually harder to comprehend than the algorithm itself. A problem here is that the untouchable variables do not appear explicitly in conventional type system specifications. They might be eliminated by unification during conventional type inference, and are notationally equal to the type the variable is unified with. Thus, such a specification requires a programmer to know how the algorithm is distributing its type variables over the program.

Lin and Sheard [2010] propose the Pointwise GADT type system. The notion pointwise comes from the correspondence between the range-type of a constructor and the type of the scrutinee in a GADT case expression. During inference in case branches, type information flows from and to such types. The authors identify two typical usage-patterns for GADTs, and make a case to restrict the use of GADTs to these patterns. *Parametric instantiation* entails that each case branch assumes the same specific instance of a polymorphic data type. *Type indexing* entails that each case branch assumes a different instance of a polymorphic data type. The authors propose a unification technique to restrict this bidirectional information flow. From an implementation point-of-view, this approach is promising: the authors performed several case studies of typical uses of GADTs and presented a mechanism that types many typical usages of GADT programs without explicit type annotations. However, it is not clear how this approach relates to the challenges we pointed out in Section 6.3.2. Furthermore, the approach lacks a specification to serve as an explanation to the programmer.

6.6.2 Type Annotation Propagation Strategies

Pottier and Régis-Gianas [2006] use shape inference. A shape represents known type information based on user-supplied signatures. These shapes are spread throughout the syntax tree in order to locate possible coercions. Their approach uses complex algorithms to spread the shapes as far as possible, iteratively. More iterations give rise to a better spread of annotations, at a cost of performance. The essential part concerning GADTs is that incompatible shapes are normalized with respect to some equation system, which is not made explicit in their work. From an implementer's point of view, this description is a concrete description of the equation system, and it requires infrastructure for spreading shapes.

Similarly, Peyton Jones et al. [2006, 2004] define a notion of wobbly types to combine type checking and type inference, which is based on earlier work on boxy types [Vytiniotis et al., 2006]. A rigid type represents known type information based on user-supplied type information, whereas a wobbly type is based on inferred information. The idea is then that type conversions are only applied on rigid types, for reasons of predictability and most-general typing. Aside from wobbly types, the authors use concepts such as "fresh most general unifiers" and lexically scoped type variables in their presentation. It is hard to distinguish which of these concepts are really required, and which of these concepts are actually related to other language features that are covered by their approach (such as type families). We incorporated a notion of wobbly and rigid types in our specification: in an explicitly typed System F-variant, the skolemnized type constants are rigid types, and we only allow type conversions on those.

The exact choice of propagation strategy is an orthogonal issue. By allowing type conversions only on known type information, a better propagation strategy just means that less explicit types have to be given by the programmer. This is the reason why we have chosen an explicitly typed source language in the first place. In fact, for our own implementation of this specification, we piggybacked on the infrastructure of the implementation of a higher-ranked impredicate type system. It has two propagation strategies, of which the advanced one has a concept of hard and soft type context, which can be compared to rigid and boxy types [Dijkstra, 2005].

6.6.3 Type Conversion Strategies

Peyton Jones et al. [2006, 2004] use a unification-based strategy, where type conversion is called type refinement. A fixpoint substitution is constructed that, for each type equation introduced by pattern matching, contains a mapping for each (non-wobbly) type component of the left hand side of an equation, to the corresponding type component on the right hand side of an equation. The substitution is then used to normalize the types under consideration. The presentation is intertwined with the type propagation strategy, which makes it hard to separate these concepts.

Wazny [2006] uses a constraint-based strategy. The GADT aspect of this strategy is covered separately by Sulzmann et al. [2006b], Stuckey and Sulzmann [2005]. They also formulate the typing problems in terms of solving constraints with CHRs. The difference is that we restrict ourselves to equality constraints, and do not need the machinery required to solve implication constraints. Their typing/translation rules do not mention how to deal with existential data types, which may be transparent to the given approach, but is of interest to a reader because GADT examples often use them. In contrast to Peyton Jones et al. [2006], restrictions on type conversions are not mentioned.

6.7 Conclusion

We presented a specification for GADT inference in terms of the language System $F_{\sim}$. This language deviates from System F in three ways: it has syntax to request an equality proof, a lambda to take an equality proof as argument and bring it in scope, and automatic coercions based on equality proofs in scope. In this language, we can express GADTs using the folklore Church encoding for data types.

Compared to other approaches, our specification is a small extension of a bare System F, which allows us to treat data types and case expressions as syntactic sugar. Furthermore, our specification describes what to do with other forms of pattern matching and binding, exploiting encodings as conventional System F terms.

As future work, it may be worthwhile to investigate if algorithms such as OutsideIn [Schrijvers et al., 2009] can be specified in a simpler way by taking an implicitly typed variant of System $F_{\sim}$ as a basis.

7 AGs with Stepwise Evaluation

Attribute Grammars are a powerful formalism to specify and implement the semantics of programming languages (e.g. as in a compiler), in particular when the semantics are syntax directed. Advanced type systems, however, have declarative specifications that encode decisions that are independent of syntax. The implementation of such decisions is hard to express algorithmically using conventional attribute evaluation.

This chapter presents Stepwise Attribute Grammars (SAGs). In a SAG, nondeterministic choices can be expressed in a natural way in conjunction with unambiguous resolution strategies based on attribute values. SAGs preserve the functional relationships between attributes and support on-demand evaluation. The exploration of alternatives are encoded as a choice between the semantic results of children. Evaluation of a child can be performed in a stepwise fashion: it is paused after each step and yields a progress report with intermediate results, until the child is reduced to its semantic value. This facilitates a breadth-first exploration of choices, until choices can be resolved based on the progress reports.

7.1 Introduction

Attribute Grammars (AGs) [Knuth, 1968] are a formalism that is particularly suited for the concise implementations for semantics of programming languages (e.g. static, operational, denotational), in the form of a compiler or interpreter. Hereby, attributes play a crucial role: properties of Abstract Syntax Trees (ASTs) can easily be expressed as attributes, and combined to form more complex properties. These attributes can be shared and additional attributes can be added on demand. For example, attributes related to name analysis and type checking can be used in a later stage for code generation or the collection of error messages. We used AGs for many small, but also several large projects, including the Utrecht Haskell Compiler (UHC) [Dijkstra et al., 2009], the Helium compiler for learning Haskell, the Generic Haskell Compiler, and the editor Proxima. AGs, and corresponding tool support, proved to be essential for these projects.

Modern programming languages allow a compiler to take some of the implementation effort away from the programmer. In C#, local type inference is employed, such that an abundance of type signatures can be omitted. Many typed OO languages have an auto-boxing feature to automatically wrap primitive values into objects. Such features save programmers from tedious tasks. To specify this freedom, programming language specification are declarative, and impose sufficient restrictions such that a deterministic algorithm exists.

Unfortunately, it is hard to deal with declarative type systems using conventional AGs. For example, it is not immediately obvious how to express Haskell's overloading for UHC using an AG. In fact, we currently rely on a constraint solver external to UHC's AGs. A lot of boilerplate code is required to interface with such a solver, and it introduces an artificial phase distinction (which hampers on-demand evaluation). Consequently, it increases code complexity severely. The goal of this chapter is therefore to extend AGs such that inference algorithms for declarative type systems can be expressed in a natural way.

Since an AG is both a functional specification and *implementation*, our challenge is to describe algorithms that resolve declarative aspects. The declarative aspects in a semantics occurs in two forms: attributes with a non-functional definition (e.g. fresh types), and productions that are not syntax-directed, but where their applicability depends on values of attributes. These two aspects are mutually expressible (Appendix 7.G). As we prefer attributes to be functionally defined, we focus on the second notion. Typically, we can deal with declarative aspects using a unification-based approach, which integrates well with AGs [Dijkstra and Swierstra, 2004]. However, some applications require a more powerful approach, with an algorithm that *actively* tries out alternatives at choice points. For example, in a Haskell compiler, to search for an equality proof for GADTs, and to resolve overloading in the presence of undecidable instances. Unfortunately, *exploration of alternatives* does not fit AGs straightforwardly.

Such explorations do not fit out of the box, because productions are selected based on the syntax (e.g. the parsed AST), and not on the values of attributes. To lift this restriction, there are approaches that generate Prolog [Walsteijn and Kuiper, 1986, Arbab, 1986]. However, we have several additional demands:

1. The approach needs to be compatible with any general purpose host language. From a theoretical perspective: to allow the extension we propose to be integrated in other AG systems as well; from a practical perspective: we have a large Haskell code base, thus want to use Haskell as a host language.

2. We need a *breadth-first evaluation* with *online results* to deal with potential infinite ASTs. Online results are also needed for integration with conventional on-demand evaluation.

3. We need complex disambiguation strategies. In general, for deterministic results and error reporting; in particular, to deal with some extensions to type classes that UHC offers [Dijkstra et al., 2007b].

In this chapter, we present an approach that adds exploration of alternatives to AGs and meets the above demands.

Our approach consists of following three key ideas:

1. We encode declarative aspects and their resolution as a *choice function* between children of a production. Overall, attributes are evaluated using conventional on-demand evaluation. Attributes of children participating in a *choice*, however, are evaluated *strictly*.

2. We annotate productions such that they *yield* user-defined progress reports that contain *intermediate results* upon strict attribute evaluation. After yielding a progress report, evaluation for a child *pauses*. This allows the choice function to evaluate its children in a *step-by-step fashion* until it has observed sufficient progress reports to commit to one of the children. It may yield progress reports itself in the mean time.

3. As implementation strategy, we map each production to a special form of coroutine that can both be run greedily until it yields the next progress report, and be run to completion directly but with a lazy result. This implementation facilitates a *hybrid evaluation model* between stepwise and on-demand evaluation, and is therefore compatible with other AG techniques that depend on the conventional on-demand evaluation.

We motivate these ideas via an example in Section 7.2.

These ideas are inspired by a parsing technique by Swierstra [2009], Hughes and Swierstra [2003] to explore alternatives based on progress information reported by parsers. The approach in this chapter is not directly related to parsing, but ultimately has its roots in the above technique (for a detailed comparison, cf. Section 7.7).

In the text, we refer to appendices published in an accompanying technical report [Middelkoop et al., 2010e] where certain topics are explored in more detail. The main contributions of this chapter are:

- We define SAGs, a language extension to Attribute Grammars that copes with declarative aspects in a semantics, while keeping the AG purely functional. We explain SAGs in Section 7.2, and show how to translate them to Haskell as a host language in in Section 7.3. We implemented this extension in the UUAG system [Löh et al., 1998].

- We introduce lazy coroutines, or *stepwise computations*, for which we provide an reference implementation (Section 7.4). In this chapter, we focus on the main ideas. The Haskell library[1] shows and explains many additional features that are useful in practice (see also Section 7.6). The library may also be used by Haskell programs that need powerful exploration capabilities, but are not related to AGs.

- As a proof of concept that these techniques are portable to other languages as well, we also provide an implementation of the example in Section 7.2 in Java (Section 7.5).

7.2 Example of a Stepwise AG for a Predicate Language

In this section, we take as running example an operational semantics[2] for a Boolean predicate language *Pred*. We show how to write *Pred*'s semantics as an AG using the notation as supported by the UUAG system. The semantics is executable: its implementation yields an interpreter in Haskell for *Pred*. Initially, the example is a functional specification, such that we can resort to a conventional AG and explain the notation involved [Löh et al., 1998]. Next we show that the semantics has an efficiency problem which we can solve by making the specification more declarative. With a Stepwise AG we then deal with it.

[1] Stepwise monad: `https://svn.science.uu.nl/repos/project.ruler.papers/archive/stepwise-1.0.2.tar.gz`

[2] For an example related to type systems, see Appendix 7.F.

7.2.1 Syntax of the Predicate Language

Consider the following grammar for the abstract syntax of *Pred*:

```
data Pred   -- Grammar for nonterm Pred as algebraic data type.
  | Var nm   :: String         -- Variable with value in Env.
  | Let nm   :: String         -- Binds (non-recursively) a
        expr :: Pred           -- name to the value of expr,
        body :: Pred           -- in scope of body.
  | And left :: Pred           -- The logical ∧ of two preds.
        right :: Pred
  | Or  left :: Pred           -- The logical ∨ of two preds.
        right :: Pred
type Env = Map String Bool     -- Environment that maps names to Booleans.
```

The data declaration introduces a nonterminal *Pred*, and a number of productions *Var*, *Let*, etc. The fields of the production comprise the symbols of the RHS of the production, consisting of a name *nm*, *expr*, etc. and a type. Some built-in types such as *Bool* and *String* specify that the symbol is a terminal, otherwise the symbol is a nonterminal.

From the grammar, we generate constructor functions *sem_Var*, *sem_Let*, etc. which are used to build attributed ASTs. Given an initial environment $\{$`"f"` $\rightarrow$ *False*, `"t"` $\rightarrow$ *True*$\}$, which binds a truth value to the free variables in the predicates, we turn a predicate into a proposition. For example:

```
taut  = sem_Or (sem_Var "t") (sem_Var "f")            -- True
contr = sem_And (sem_Var "t") (sem_Var "f")           -- False
alias = sem_Let "a" (sem_Var "t") (sem_Var "a")       -- True
big₁  = sem_And (sem_Var "f") big₁                    -- False
big₂  = sem_And big₂ (sem_Var "f")                    -- False
```

Informally speaking, the Boolean value of *taut* is *True* and of *contr* is *False*. The big_1 and big_2 predicates are large sequences of *False* that are combined with *and*s. In fact, these sequences are infinitely long, although we only use that for emphasis. Their truth value is *False*. Formally, however, we have to define an operational semantics to make such statements.

7.2.2 Deterministic Operational Semantics

An operational semantics for a predicate takes an environment and provides a truth value. We model these two aspects as attributes, which correspond to values attached to the nodes of an AST: an *inherited* attribute *env* that represents the environment for the subtree, and a *synthesized* attribute *val* that denotes the value of the subtree (for the given environment):

```
attr Pred inh env :: Env syn val :: Bool   -- Typed attributes for nonterm Pred.
```

The obligation to define an inherited attribute for a node in the AST lies with the parent, for a synthesized attribute it lies with a child. Via a **sem**-block, we define for each production, the production's synthesized attributes, and the inherited attributes of its children using rules written in the AG's host language (in our case: Haskell expressions). These rules may refer to the inherited attributes of the production and the synthesized attributes of the children. We refer to an attribute using *chld.atname* notation. To refer to the attributes of the production itself, we use the special name *lhs*. We refer to terminals by their name. Thus, we define the semantics for predicates as:

```
sem Pred   -- Specifies rules for productions of nonterm Pred.
  | Var  lhs.val   = lookup nm lhs.env
  | Let  lhs.val   = body.val    -- Takes val from child body.
         body.env  = insert nm expr.val lhs.env
  | And  lhs.val   = left.val ∧ right.val
         left.env  = lhs.env      -- Copies down env to the left.
         right.env = lhs.env      -- Copies down env to the right.
  | Or   lhs.val   = left.val ∨ right.val
         left.env  = lhs.env
         right.env = lhs.env
```

Thus, we simply pass down the environment from top to bottom. For a variable, we lookup the associated value in the environment. For a let-binding, we insert the value in the environment. Finally, for the *And* and *Or*, we take the Haskell (short-circuiting) $(\wedge)$ and $(\vee)$ on Boolean values of the children.

From this AG, the UUAG compiler generates an interpreter that takes a predicate, defines the root's *env* attribute, and demands a value for root's *val* attribute. On-demand evaluation proceeds to compute those attributes when their values are needed during the computation of the *val* attribute. In a purely functional language such as Haskell, we can represent a decorated tree as a function from inherited to synthesized attributes [Saraiva and Swierstra, 1999]. More precisely, for each production (e.g. *Var*, *And*), we generate a function (*sem_Var*, *sem_And*, respectively) that, given the functions corresponding to its children, represents a Haskell function takes values for inherited attributes as parameter, and returns a product with values for the synthesized attributes (Section 7.3).

When we run the interpreter on the examples given earlier, it gives the expected outcomes, with a single exception: the computation for big_2 diverges. The reason is that both $(\vee)$ and $(\wedge)$ start with the left argument first. This may involve a lot of computation (e.g. in case of big_2) that could be avoided by looking at the second argument first. Then, however, big_1's evaluation would diverge. If big_2 would be a long but finite sequence, then its evaluation would not diverge but take a long time to produce an answer.

7.2.3 Declarative Operational Semantics

The $\wedge$ and $\vee$ operators do not distribute evaluation over their operands well. To give more freedom with respect to the evaluation order, we could add non-determinism to the specification, for example via multiple (conditional) interpretations of a production, using the following (fictional) notation:

```
sem Pred     -- Productions with conditional alternatives.
   | And₁ lhs.val = left.val      when left.val   = False
   | And₂ lhs.val = right.val     when right.val = False
   | And₃ lhs.val = left.val      when otherwise

   | Or₁  lhs.val = left.val      when left.val   = True
   | Or₂  lhs.val = right.val     when right.val = True
   | Or₃  lhs.val = left.val      when otherwise
```

In this specification, there is freedom in the alternative to apply. A clever AG evaluator could use a breadth-first exploration of alternatives combined with prioritizing those attributes that are closer to the root and thus provide a more balanced evaluation strategy.

However, we want to be precise in this strategy. When both *left.val* and *right.val* are *True*, the choice between Or_1 and Or_2 is ambiguous. For predictability reasons (and *referential transparency*), it must be clear which one should be taken; also, we may prefer one over the other based on some other available information. In some cases we may want to be biased towards a particular subtree, potentially based on results computed at runtime. Hence, as mentioned in Section 7.1, we want to be able to define this strategy ourselves.

7.2.4 Stepwise Operational Semantics

To define such a strategy, we take a less ad-hoc solution. In particular, we encode alternatives as a choice between children instead of productions. We do not loose expressiveness (Appendix 7.E), and gain the ability to define strategies in terms of the evaluation of children. In the remainder of this chapter, we focus on *Or*-predicates only, and leave the strategies for *And*-predicates to the reader. For the *Or*-predicate, if we know that one of the children's *val* attribute evaluates to *True*, we can commit the choice to that child. We express this as follows, using a function $choose_{or}$ (to be defined later):

```
sem Pred | Or    -- Rules for production Or of nonterm Pred.
   left.env   = lhs.env                  -- Copies down env to the left.
   right.env = lhs.env                   -- Copies down env to the right.
   merge left right as res = choose_or   -- Creates child res using choose_or to merge.
   lhs.val    = res.val                  -- Pass val up from child res.
```

The merge-rule transforms children *left* and *right* into a single virtual child *res*. We may refer to the synthesized attributes of *res*, but not to those of *left* and *right*. Similarly, we need to define the inherited attributes of *left* and *right*, but may not define those of *res*.

The function $choose_{or}$ takes the synthesized attributes of *left* and *right* as arguments, and is required to provide values for the synthesized results for *res*[3]. As initial attempt, we define $choose_{or}$ as:

```
choose_or :: Bool → Bool → Bool
choose_or left_val right_val   -- Synthesized attributes of left and right as parameter.
```

[3]Due to Haskell's laziness, on-demand evaluation starts as soon as we scrutinize a value of an attribute.

```
| left_val   = left_val    -- Takes the left value, or
| right_val  = right_val   -- takes right value, or
| otherwise  = left_val    -- otherwise takes the left value.
```

With this function, we unambiguously specify what kind of solution we want. However, in terms of evaluation we are back where we started: we evaluate the entire left child first, thus evaluation is still left biased.

Scrutinizing the value of a synthesized attribute leaves us little control over the scheduling of the evaluation[4]. In the example, we cannot return a result until we make the choice, but in order to do so, we need to inspect the result, which causes one of the children to evaluate fully before we have a chance to inspect the other one. Instead, we want the scheduling decisions to be based on explicitly indicated intermediate results. On-demand evaluation does not help us here, and therefore we propose a different evaluation scheme to choose between alternatives.

This leads us to the second idea of this chapter: we evaluate a child under a choice strictly (instead of on-demand), thus computing attributes of children in a fixed order[5]. Instead of completely evaluating all synthesized attributes of such child, however, we evaluate it just far enough to yield a progress report. It then suspends to be resumed later. We explain later how to emit such progress reports during evaluation. To explore or merge two children both gradually and simultaneously, as well as report intermediate results to the parent, we take their progress reports alternatively, and intertwine them (explained below).

In our example, we wish to prioritize evaluation to the child that we estimate has performed the least amount of work, to balance out evaluation. For that, the progress reports need to give an indication that some work has been done. Hence, we define a type for progress reports (e.g. *Info*), with a plain constructor *Work* as possible value:

```
data Info = Work    -- Application-specific type defined by programmer.
```

The value for a child that is passed to the $choose_{or}$ function is not just the value of the synthesized attribute, but now a *stepwise computation* of the type *Stepwise Info Bool*, where *Info* is the type of the progress reports, *Bool* is the type of the synthesized attribute. The type of $choose_{or}$ changes to:

```
choose_or :: Stepwise Info Bool → Stepwise Info Bool
                → Stepwise Info Bool
```

A computation of the type *Stepwise Info Bool* can be manipulated with a simple monadic API (Section 7.4):

```
emit      :: i → Stepwise i ()          -- Type i equals Info here.
smallStep :: Stepwise i α → Report i α  -- Evaluates to next report.
```

4 We can use Haskell's lazy evaluation to return lazy approximations of the final values as result. This, however, complicates the AG severely, as rules must be manually lifted to operate on approximations.

5 In this chapter, we take the order of appearance of children in a production as strict evaluation order. UUAGC actually supports Ordered Attribute Grammars (OAGs) [Kastens, 1980] that takes attribute dependencies into account, and ensures that all attributes are well-defined (non-cyclic). Using OAGs, the order of appearance is not important.

```
data Report i α = Done α | Step i (Stepwise i α)   -- Result of smallStep.
return :: α → Stepwise i α
(≫)  :: Stepwise i α → Stepwise i beta        → Stepwise i α
(≫=) :: Stepwise i α → (α → Stepwise i α) → Stepwise i α
```

Function *emit* yields a progress report, and *smallStep* evaluates the computation until it is *Done* (with attribute values) or yields a *Step* (with continuation). Computations are composable via monadic operators. The monadic sequence ($m_1 \gg m_2$) performs m_1, throws away its result, then performs m_2 and delivers m_2's result. The monadic bind $m \gg\!= f$ performs f parameterized with the result of m.

Using the above API, we now redefine $choose_{or}$ as:

```
choose_or left right = choose′ (smallStep left) (smallStep right) where
  choose′ (Done v) _ = if v then left  else right   -- Choose.
  choose′ _ (Done v) = if v then right else left    -- Choose.
  choose′ (Step Work left′) (Step Work right′)       -- Both yielded a Work.
    = emit Work ≫ choose_or left′ right′             -- Pass on the report.
```

Both children perform a step. When one of them evaluates to *Done*, we inspect its attribute and make a choice. By replacing the expression with either *left* or *right*, we eliminate the other choice. Otherwise, we emit a step that a bit of work has been done (for the current node), and retain the choice between the continuations of the children. This strategy effectively encodes a breadth-first exploration of the children.

We emit a *Work* progress report for each *Var* node. To inject such reports, we extend the *Var* production with a special built-in nonterminal *Progress*[6][7]. This nonterminal has a single inherited attribute named *info*, and no synthesized attributes. Since it has no synthesized attributes, a child of this nonterminal is never evaluated during on-demand evaluation. During strict evaluation, however, each child is evaluated. In that case, the implementation of *Progress* yields the progress report that it took as parameter[8]. This is exactly the behavior that we want. We are not interested in progress reports during on-demand evaluation, but the more we are interested in them during strict evaluation.

```
data Pred | Var  report : Progress     -- Additional child of production Var.
sem  Pred | Var  report.info = Work    -- Defines its inherited attribute.
```

The *info* attribute is defined by a conventional rule, thus as right-hand side, we have access to any intermediate result (not needed for this example).

[6] Alternatively, the additional child can be specified as a higher-order attribute using Higher-Order AGs [Vogt et al., 1989] (supported by UUAG), which does not require us to change the production.

[7] Actually, the *Progress* nonterminal can be implemented using a merge-rule (Appendix 7.A).

[8] Visits to children may be omitted if the synthesized attributes of that child are not used. This is undesirable when the child may emit a progress report. In Appendix 7.A, we explain that progress reports themselves can be considered to be a hidden attribute. Hence, visits to children that emit progress reports can never be omitted during stepwise evaluation. Also, referential transparency is preserved.

7.2.5 Hybrid On-demand and Stepwise Evaluation

The attribute rules are oblivious towards stepwise evaluation. They are still pure functions between attribute values, which is important to reason with AGs. Access to progress reports is only possible in merge functions. Additionally, this allows stepwise and on-demand evaluation to coexist.

Stepwise evaluation is strict, therefore it cannot deal with attributes defined in a cyclic way. However, we only need stepwise evaluation while making a choice: once only one alternative is left, we can continue with on-demand evaluation. For that purpose, we provide a function:

$$\mathit{lazyEval} :: \mathit{Stepwise}\ i\ \alpha \rightarrow \alpha \quad \text{-- Runs computation lazily, returns syn attributes.}$$

It takes a partially reduced child as parameter, ignores progress reports, and returns the attributes (α is instantiated to a product of the attribute types) on which we may perform conventional on-demand evaluation. Further details for *lazyEval* can be found in Section 7.4.

The global picture is that we start with *lazyEval* at the root of the AST. When *lazyEval* needs an attribute value of a merged child, it asks for the result of the merge function (e.g. choose_{or}). Consequently, choose_{or} invokes *smallStep* on its children, gradually reducing the candidate children, and ultimately returns one of the residual children (e.g. *left*). Then *lazyEval* proceeds with this child.

To implement these ideas, we arrive at the third idea of this chapter: The AG is compiled to a special form of coroutines (Section 7.3). A coroutine is a function that may yield an intermediate result and then suspends. Its caller receives that result, and can reinvoke the coroutine again to resume its execution. This gives us the behavior of *smallStep*. For *lazyEval*, we need some special behavior: in that case, a coroutine resumes and runs immediately towards its end. It constructs only the administration needed to perform the remaining computations in an on-demand fashion. We give an implementation for these special coroutines in Section 7.4.

7.3 SAG Translation

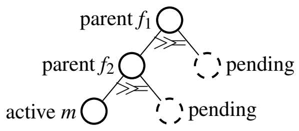
A SAG is a conservative extension of an AG that adds the **merge** rule. We sketched its static semantics in the previous section. In this section, we sketch its denotational semantics: we describe how to map a SAG to a monadic Haskell program (Translation scheme in Appendix 7.B). The monad is defined in Section 7.4. The SAG translation is largely based on a conventional translation to Haskell, as implemented in UUAG [Saraiva and Swierstra, 1999].

To translate a SAG, we translate each production to a coroutine (e.g. *sem_And*), implemented as a monad. With the coroutines we build an attributed tree (e.g. *taut* and big_1 in Section 7.2.1). This tree is represented as a *function* from the inherited attributes to a product of the synthesized attributes (and, as mentioned in the previous section, lifted in the stepwise monad). We call this tree the *semantic value* or simply the *semantics* (of the associated nonterminal/production). Thus, a coroutine is a function that takes the *semantic values* (or, simply: *semantics*) of its children as parameter, and returns its own semantics. For example, in case of nonterminal *Pred*, the type of its semantics (named *I_Pred*) and the signature of *sem_And* are:

```
type I_Pred = Env → Stepwise Info Bool   -- Function type for attributed tree.
sem_And :: I_Pred → I_Pred → I_Pred      -- Coroutine for And production.
```

We first discuss a translation of conventional AGs (*sem_And* does not use **merge**). Coroutine *sem_And* takes the semantics of its children as parameter. It must return its own semantics, which is a function from its single inherited attribute *env* to its single synthesized attribute *val* lifted in the stepwise monad. We use the encoding *childXattr* to unambiguously refer to an attribute, where *X* equals *I* for an inherited attribute and *S* for a synthesized attribute. The function body—the monad—comprises the *plan* for a production: it consists of the calls to the children, and definitions of the attributes. At a function call to a child, we pass its inherited attributes, then we obtain a monadic value with the synthesized attributes, which we can match against. Recursive do-notation [Erkök and Launchbury, 2000] allows us to write such plans concisely.

```
sem_And left right = λlhsIenv → do rec   -- Takes inh attr env.
   leftSval  ← left leftIenv              -- Calls left child.
   rightSval ← right rightIenv            -- Calls right child.
   let leftIenv  = lhsIenv
   let rightIenv = lhsIenv
   let lhsSval   = leftSval ∧ rightSval
   return lhsSval
```



Under the hood, the do-notation reorders the statements to produce a linearized plan as a sequence of monadic binds[9], e.g. (let-bindings omitted, replaced by dots):

```
left leftIenv ≫= (λ... → right rightIenv ≫= (λ... → return lhsSval))
```

A monadic bind $m \gg= f$, expresses that the remainder of the plan f is parametrized with the results of plan m (see the figure above). Strict evaluation goes from left to right, thus reducing a plan gradually.

To translate the merge rule, consider the translation for *sem_Or*. The calls to the involved children are not made part of the plan: we do not match against their results. Instead, $choose_{or}$ takes the plans (stepwise computations) of its input children, and as result, must provide a plan (computation) for its output child. Subsequently, we match against the output child to obtain the synthesized attribute *res.val* of merged child *res*:

```
sem_Or left right = λlhsIenv → do rec   -- Function from lhs.env to lhs.val.
   resSval ← choose_or (left leftIenv) (right rightIenv)   -- Translation for merge.
   let leftIenv  = lhsIenv   -- Defines env for left.
   let rightIenv = lhsIenv   -- Defines env for right.
   let lhsSval   = resSval   -- Takes val attribute from res child.
   return lhsSval            -- Returns result for Or production.
```

[9] In general, also calls to *mfix* are inserted to deal with cyclic definitions. We provide a definition for *mfix* in the library associated to this chapter. In practice, with UUAG, we use Ordered Attribute Grammars, that result in a more sophisticated translation that only needs non-recursive do-notation, without *mfix*.

SAGs are thus only a modest extension to the conventional translation. Most of the additional complexity is hidden in the implementation of the coroutine (Section 7.4).

7.4 Lazy Coroutines and the Stepwise Monad

We use a coroutine to represent the residual attributed tree after strict evaluations. We either transform this coroutine to its lazy result via *lazyEval*, or run it greedily using *smallStep* to to the point where it yields the next progress report. The data type *Stepwise i a* represents such a coroutine. It exposes the following structure:

```
data Stepwise i a where        -- Stepwise is a ‘defunctionalized’ monad.
  Yield   :: i → Stepwise i a → Stepwise i a   -- Paused computation
  Fail    :: String           → Stepwise i a   -- Aborted computation
  Return  :: a                → Stepwise i a   -- Finished computation
  Pending :: Stepwise i b → Parents i b a → Stepwise i a
data Parents i a b where       -- Parent stack (root of type b, active child a).
  None :: Parents i a a        -- Bottom of the stack.
  Bind :: (a → Stepwise i z) → Parents i z b → Parents i a b
```

Yield means that the coroutine paused to yield a progress report of the type *i*. The second component represents the continuation. *Fail* represents an aborted computation, and *Return* means it succeeded, providing a value of type *a*.

During strict evaluation, reduction of a child starts only when its preceding sibling is fully reduced (linear execution of the plans in Section 7.3). Hence, every node has at most one child active, and a continuation of what to do after that child is finished. A *Pending* value encodes this: the first component is the deepest child awaiting further evaluation. The second component is the stack of parent-continuations. The GADT *Parents i a b* represents the parent nodes that await a value of type *a*, to ultimately compute a value of type *b*. When we match *None*, we reached the bottom of the stack, where we ensure that *b* equals *a*.

We can now give a *Monad* instance for *Stepwise*. The bind $m \gg\!= f$ is encoded as active child *m* with single pending parent *f*:

```
instance Monad (Stepwise i) where     -- Via the monad combinators, we
  return  = Return                    -- thus build a Stepwise-value, and
  fail    = Fail                      -- reduce this value via
  m >>= f = Pending m (Bind f None)   -- smallStep or lazyEval.
emit i = Yield i (return ())
```

Given a coroutine, we can run it immediately to its lazy result value. This process describes how we transform a residual tree into a tree on which we can perform on-demand evaluation. We step over any progress reports it may yield in the process. When we encounter a *Pending*, we apply *lazyEval* to get a lazy result of the child, and use *evalPending* to provide it to its parent, which in turn passes it to its parent, until we reach the root:

```
lazyEval :: Stepwise i a → a                     -- Interprets computation lazily.
lazyEval (Yield _ m)     = lazyEval m            -- Skips progress report.
lazyEval (Fail s)        = error s               -- Interpreted as ⊥.
lazyEval (Return v)      = v
lazyEval (Pending m p) = evalPending p (lazyEval m)

evalPending :: Parents i a b → a → b            -- lazyEval on chain of parents.
evalPending None       a = a                    -- Reached the root.
evalPending (Bind f r) a = evalPending r (lazyEval (f a))
```

Given a coroutine, we can also run it greedily until it yields the next progress report. Either it fails, is finished, or is paused with the yielded information *i* and the continuation to resume it with:

```
data Report i a where                            -- Outcome of smallStep.
  Failed :: String              → Report i a     -- Aborted with message.
  Done   :: a                   → Report i a     -- Finished with value.
  Step   :: i → Stepwise i a → Report i a       -- Paused with user report.
```

The function *smallStep* performs a strict reduction. Once it encounters a *Yield* it can stop and return a *Step*:

```
smallStep :: Stepwise i a → Report i a          -- Evaluate until next report.
smallStep (Yield i m)     = Step i m             -- Pause after a yield.
smallStep (Fail s)        = Failed s
smallStep (Return v)      = Done v
smallStep (Pending m p) = reduce m p            -- Continues with m, and possibly p.
```

For a *Pending*, its result depends on the reduction of the active child. If it finishes without yielding a progress report, we pass the result to its parent and continue reducing the parent. If the active child itself turns out to be a *Pending*, we push its stack on the stack we already have, and continue reduction:

```
reduce :: Stepwise i a → Parents i a b → Report i b
reduce (Yield i m) r          = Step i (Pending m r)   -- Keeps residual parents.
reduce (Fail s) _             = Failed s
reduce m None                 = smallStep m            -- No parents left.
reduce (Return v) (Bind f r) = reduce (f v) r          -- Proceeds with parent f.
reduce (Pending m r′) r      = reduce m $ push r′ r   -- Concatenates stacks.

push :: Parents i a b → Parents i b c → Parents i a c
push None r         = r                        -- Appends to the bottom.
push (Bind f r′) r = Bind f (push r′ r)       -- Walk towards the bottom.
```

The merging of the parent stacks is important. The stack represents a whole subtree, with the active child on top, and the root of the subtree on the bottom. When we want to reduce this subtree one step, we can thus immediately reduce the active child without having to traverse through the parents.

We provide the API that we discussed in this section as a Haskell library. Its implementation satisfies the monad laws (Appendix 7.H) for both *lazyEval* and *smallStep* evaluation. Furthermore, if it holds that $\mathit{smallStep}^{*}\ m = \mathit{Done}\ v$ then also *lazyEval* yields this value $\mathit{lazyEval}\, m = v$. The converse is not necessarily true: $\mathit{lazyEval}\ (\bot \gg \mathit{return}\ ()) = ()$, whereas $\mathit{smallStep}^{*}\ (\bot \ggg \mathit{return}\ ()) = \bot$. However, when the AG is ordered, and each rule well-founded, then also the converse holds.

7.5 Imperative Implementation

The reference implementation that we presented in the previous section relies on several Haskell features, such as lazy evaluation, monads and GADTs. Despite that, the approach is portable to imperative languages and AG systems for these languages. As a proof of concept, we ported the example of Section 7.2 to Java (Appendix 7.J), and implemented a small Stepwise support library[10].

We encode demand-driven AGs as conventional for object oriented languages. AST nodes are represented by objects, using subclasses for productions. Attributes are encoded as fields on such nodes using getters and setters together with lazy initialization. We map each rule to a *Runnable* object. A rule is associated with one or more attributes. When the value of an attribute is not yet defined, the associated rule is executed, and the rule defines these attributes using side effect. A rule may refer to the values of other attributes, thus driving on-demand evaluation. Executing a rule twice has no effect.

Additionally, a node exposes a visit method that encodes the coroutine for strict evaluation. The visit method may be invoked multiple times, and returns either with progress information, or a pointer to a child node to evaluate first, or indicates that evaluation is done for the subtree. With each execution, the visit method executes some of the rules. Nodes can thus be evaluated strictly via the visit method, and on-demand by accessing the attributes directly.

As discussed in Section 7.3, children of a production come in two fashions: conventional children and merged children. Conventional children are conventional AST nodes, which expose inherited attributes. Merged children do not expose inherited attributes. To start evaluation and access the synthesized attributes from both types of children, a stepwise computation can be requested from a child. A stepwise computation is a coroutine-object that supports the *lazyEval* and *smallStep* operations of Section 7.4. It represents the evaluation of the subtree rooted by the child.

The stepwise computation obtained from a conventional child x represents the stack of nodes under strict evaluation. A node on the stack has been partially evaluated strictly and is waiting for strict evaluation of nodes above it to complete before proceeding with strict evaluation. The child x is on the bottom of the stack. The active child is the top of the stack. For the *lazyEval* operation, the stepwise computation obtained from a conventional child directly returns x. This corresponds to the *evalPending* operation, except that in contrast to the functional implementation, we do not have to thread the lazy outcomes around because these are represented by pointers and side effect in the imperative implementation. For *smallStep*, strict

[10] Stepwise Java example: `https://svn.science.uu.nl/repos/project.ruler.papers/archive/jstepwise.jar`

evaluation proceeds with the active node. If it yields progress info, then these are returned as the report for *smallStep*, and the evaluation is effectively paused again. If evaluation of the active node is finished, the node is popped from the stack and evaluation continues with its parent. If an active node requests evaluation of a child first, these are pushed on the stack.

The stepwise computation obtained from a merged child is an object with a visit method that has access to the stepwise computation of the children that are being merged. With each visit, it must return a progress report. In particular, it can report that the computation is to be replaced with a computation of the children, thereby resolving the choice. Essentially, such a computation is an imperative version of the monadic $choose_{or}$.

The imperative implementation closely resembles the functional implementation, although more verbosely. A language with native support for coroutines would simplify the implementation slightly.

7.6 Remarks

7.6.1 Extensions

We used SAGs to rapidly prototype a type-directed transformation. It required a small extension of the presented ideas: *semantic lookahead* (Appendix 7.C). Often, a choice for a subtree has consequences at another location in tree. We thus (may) need to investigate potential alternatives beyond the evaluation of the subtree. To this end, we added a mechanism to obtain the continuation, and investigate the steps coming out of the continuation.

To combine progress reports of different types, and allow them to depend on the result type of the computation itself, we implemented *transcoding* (Appendix 7.D). It can also be used to replace multiple reports by a single report (compression) to trade interleaving granularity with fewer reports to examine.

Finally, we offer *explicit sharing* via references. We use this mechanism to deal with the *MonadFix* instance required for recursive do-notation, and also to offer *memoization* (to turn the AG under user-defineable conditions into a graph).

7.6.2 Benchmarks

We benchmarked our approach (Appendix 7.I) on a standard MacBook 2.1 with a 2 GHz dual-core processor, 2 GB of main memory, and GHC 6.12.1. We compared the execution time of code generated the conventional way by UUAG against code that uses stepwise evaluation, and meassured the runtime overhead. The throughput of stepwise binds is about thirty times slower than the bind of the identity monad. The overhead is constant for *nextStep*, and is a bit more erratic for *lazyEval* (but comparable). On the other hand, the overhead is negligible in practice. We compared the compilation speeds of UHC, which makes heavy use of AGs (for both large and small tasks). The compilation time only marginally increased, and stays under random noise induced by garbage collection.

7.7 Related Work

This chapter is heavily inspired by uu-parsinglib [Swierstra, 2009, Hughes and Swierstra, 2003]. The parsing library supports both context-free and monadic grammars, and offers online results as well as error correction. It offers a data-type similar to our *Stepwise*. The essential difference is that uuparsing-lib's bind cannot yield a result until the LHS of the bind is fully step-wise evaluated[11]. Instead, required for the hybrid evaluation model, our implementation can yield a result when the RHS can do so (when using *lazyEval*). Also, uuparsing-lib's implementation manually manages stacks with semantic values computed so far (the outcome of the history parser), or semantic values still to come (the outcome of the future parser). Instead, in our implementation, these values are implicitly represented as local variables in closures.

Our work is related to various disambiguation strategies. Some approaches allow ambiguous ASTs and impose syntactic restrictions to resolve these, e.g. by conditionally rejecting certain productions based on the AST structure, or prioritizing some productions over others [van den Brand et al., 2002]. Other approaches generate a parse forest and filter later, potentially using semantic information [Bravenboer et al., 2005]. In contrast, our approach does not require all alternatives to be available a priori, and works for also in case of infinite trees and in combination with nonterminal attributes.

AGs are traditionally considered to select productions deterministically based on the syntax of a language. Conditional Attribute Grammars [Boyland, 1996], as supported by the FNC-2 system [Jourdan and Parigot, 1991] and our experimental Ruler system [Middelkoop et al., 2010a], allow multiple definitions for a productions guarded by conditions. These conditions need to be evaluated first, thus offer limited control over the exploration of alternatives.

There are AG evaluators that generate Prolog code [Walsteijn and Kuiper, 1986, Paakki, 1991]. Such an approach that depends on a logic language is unacceptable for us, as it does not mix well with our existing Haskell code. In contrast, our approach can be implemented in an arbitrary general purpose host language. The lazy evaluation our Haskell implementation relies on, actually just represents on-demand evaluation that other AG approaches provide. Moreover, as sketched by Section 7.2.5, our approach is compatible with circular and remote AGs [Magnusson and Hedin, 2007].

There are several different techniques to deal with declarative aspects in the specifications of programming languages. We classify declarative sources in increasing complexity:

- *Deterministic*. Both the production selection and the value of attributes are purely functional. A problem in this class trivially fits an AG. For example, auto-boxing and implicit coercions fall in this category.
- *k pass*. A priori unknown values (type variables) and derivations (deferred judgments) are replaced with place holders. Nodes in the AST are traversed at most *k* times. A

[11] Formally: we can write a conventional AG as a lazy applicative functor. Monads are more expressive, hence we require the following equality to hold, which is not the case for uu-parsinglib:

$$p \star q \quad \equiv \quad p \gg\!= \lambda f \to q \gg\!= a \to \mathit{return}\ (f\ a)$$

decision about a place holder must be taken during one of these traversals. Typically, information about place holders is maintained in a substitution (concrete assignments to variables) or constraints (symbolic representation of a deferred judgment). For example, Hindley-Damas-Milner type inference (algorithm W) has $k = 1$. After one traversal, a type variable either got assigned a concrete type, or it is fixed by a skolem constant (and generalized later). Haskell 98 overloading resolving is an example of $k = 2$. In the first pass, type equalities are resolved and class-constraints are collected. In the second pass, the class-constraints are resolved. Such problems can be dealt with in AGs using additional attributes for substitutions and constraints.

- *ω pass*. Declarative aspects that are resolved by fixpoint iteration falls in this class. This includes the class of type and effect systems. Also, resolution of Haskell's overloading in combination with functional dependencies falls in this class. AGs with circular references can be used to encode such problems [Magnusson and Hedin, 2007, Jones, 1990], or AGs that can express iteration [Middelkoop et al., 2010a].

- *Exploration of alternatives*. In the previous classes, declarative aspects are resolved by sufficiently constraining a value, where the constraints are a pure function of the (attributed) AST. In the exploration class, declarative aspects are resolved by exploring assignments to place holders. This requires instantiations for place holders to be enumerable. Haskell's overloading combined with overlapping instances, and the construction of equality coercions for GADTs fall in this class. This chapter positions itself in this class.

- *Undecidable*. Inference for some declarative aspects is undecidable. For example, a polymorphic type can in general not be inferred for an argument of a recursive function.

Haskell offers several library approaches for backtracking, via folklore *Maybe* and list monads to more advanced monads [Hinze, 2000, Kiselyov et al., 2005, Fischer et al., 2009] that deal with nondeterminism and lazyness. Alternatives are only explored for a value when this value is required. However, the order of appearance of alternatives affect memory retainment and how online the results are. See the discussion in Swierstra [2009, Section 4.1].

Coroutines were considered for many compilation tasks [Marlin, 1980]. Nowadays, they are mostly used to implement producer-consumer patterns. Kastens [1980] showed how to compile multi-visit AGs to coroutines. In this chapter we apply coroutines in a different fashion. We use coroutines to expose explicitly indicated parts of the internal state of an AG evaluation, in order to describe exploration strategies.

There are several approaches for monadic coroutines in Haskell. These implementations have their roots in the folklore CPS monad to pause, abort and merge computations. Kiselyov's Iteratees [Kiselyov, 2008] come close to our implementation. However, technical differences aside, there is a conceptual difference. Typical coroutine implementations allow invocations to take additional arguments. Since the result may depend on the value of such a parameter, a lazy result cannot be given until the last invocation. Therefore, the hybrid evaluation model that we presented cannot be implemented with such coroutines.

7.8 Conclusion

We presented SAGs, a powerful language extension to Attribute Grammars to cope with declarative aspects in the semantics of programming languages. We stated our requirements in Section 7.1, and showed how our approach meets these demands by example in Section 7.2, and sketched the implementation in Section 7.3 and Section 7.4. We implemented SAGs in the UUAG system.

The idea central to our approach is to encode alternatives as a choice between children of a production, and resolve this based on stepwise inspection of intermediate results of these children in the form of progress reports.

As future work, we intend to replace the overloading mechanism as currently implemented in UHC using the new AG features as presented in this paper. Also, a remaining question is if the stepwise monad is powerful enough to simplify the implementation of `uu-parsinglib`.

In Appendix 7.A, Appendix 7.E, and Appendix 7.B we go into more detail of the AG part of the story. Appendix 7.C and Appendix 7.D show improvements of the stepwise monad. Appendix 7.F gives an example that makes use of the additional improvements. Appendix 7.G shows how various declarative aspects can be expressed in terms of each other. Proofs for some of our claims related to the monad laws can be found in Appendix 7.H, and benchmark results in Appendix 7.I. Finally, we show a translation to Java in Appendix 7.J. The thesis contains appendices A-D. The extended edition contains the remaining appendices.

7.A Progress Reports and their Emission

As we mentioned in Section 7.2, we annotate productions with the built-in nonterminal *Progress* to yield progress reports. Actually, this built-in nonterminal is only a notational convenience: it is definable in terms of the merge-syntax that we presented before. In this section, we show the implementation.

The data type *Progress*, as mentioned in this chapter, can be implemented as follows:

```
data Progress | Progress
attr Progress inh info :: a
sem Progress | Progress
  merge as res : Progress = emit lhs.info ≫ return ()
```

It has a single inherited attribute, and a single production. This production has a single child *res*, merged out of an empty set of children. To define the semantics of this child, we thus do not get any semantics of children as parameter. Since *Progress* does not have any synthesized attributes, defining a semantics for *res* is straightforward: we return the empty tuple ().

It is not immediately clear why this implementation would work: nonterminals without any synthesized attributes never need to be visited. Referential transparency tells us that we may replace a child with a product of its synthesized attributes, and attribute references with the corresponding values. A child without synthesized attributes may never be evaluated, and the progress report never yielded.

The essential realization here is that there is that a hidden attribute plays a role: the progress reports themselves are a purely functional attribute. Hence, during strict evaluation via *smallStep*, we still visit children without any explicitly declared synthesized attributes in order to get the progress reports. In contrast, during on-demand evaluation via *lazyEval*, we ignore progress reports, hence do not evaluate children without synthesized attributes.

7.B Translation Scheme

In this section, we formalize SAGs. We first define a small core language *sagcore*, consisting of Haskell extended with embedded AG blocks, obtained by desugering AG descriptions. The following grammar lists the syntax of these embedded AG blocks:

$i ::= \mathbf{attr}\ I\ \mathbf{inh}\ \overline{a1}\ \mathbf{syn}\ \overline{a2}$	-- attribute delcs
$a ::= x :: hty$	-- attribute decl, with Haskell type *hty*
$s ::= \mathbf{sem}\ I\ \overline{r}$	-- semantics expr, defines production for *I*
$r ::= p = e$	-- binds to *p* to pure *e*
$\mid\ \mathbf{child}\ c :: I = e$	-- declares child
$\mid\ \mathbf{merge}\ \overline{c}\ \mathbf{as}\ c :: I = e$	-- declares merged child
$o ::= x.x$	-- expression, attribute occurrence
x, I, p, e	-- identifiers, patterns, expressions respectively

Attribute declarations *i* declares all attributes (name and type) of a nonterminal *I*. A semantics block defines a single production for a nonterminal *I*, and gives its rules $\overline{r}$. Productions in *sagcore* are nameless: we use a Haskell declaration to give it a name. Furthermore, we declare its children through rules. Rules either define attributes, or declare children: we introduce all children as higher-order attributes (Appendix 7.E). The expressions *e* are Haskell expressions, with possible attribute occurrences *o*. Patterns *p* are Haskell patterns, also with possible attribute occurrences *o*.

For example, we show how the production *Or* as mentioned in Section 7.2 is encoded in *sagcore*. In UUAG-notation, we declare a production *Or*, declare the attributes of the corresponding nonterminal, and give the rules for the attributes:

```
data Pred | Or left, right :: Pred
attr Pred   inh e :: Env   syn b :: Bool
sem Pred | Or
  left.e  = lhs.e
  right.e = lhs.e
  merge left right as res :: Pred = choose_or
  lhs.b   = res.b
```

In *sagcore*, we declare the attributes of the nonterminal, then use a Haskell function to represent the production: it takes the semantics of the children as parameter, then uses an embedded semantics block to define the semantics for the production itself:

```
**attr** $Pred$ **inh** $e :: Env$ **syn** $b :: Bool$
$sem\_Pred\_And\ l\ r =$
  **sem** $Pred$
    **child** $left :: Pred\ \ = l$
    **child** $right :: Pred = r$
    **merge** $left\ right$ **as** $res :: Pred = choose_{or}$
    $left.e\ \ = lhs.e$
    $right.e = lhs.e$
    $lhs.b\ \ \ = res.b$
```

We thus keep the rules, yet express the grammar directly as Haskell functions:

$$\begin{array}{lll} \textbf{data}\ N \mid C\ \overline{c :: I} & \rightsquigarrow & sem\_N\_C\ \overline{c} = \textbf{sem}\ N \\ \textbf{sem}\ N \mid C\ \ \overline{r} & & \quad \overline{\textbf{child}\ c :: I = c}; \overline{r} \end{array}$$

With such a function that represents a production, we can construct attributed trees. Each node in the tree has its own set of inherited and synthesized attributes: the associated nonterminal specifies their name and their types. The rules of the production define the attributes of the production, and declare what attributes the children have. The production must define its synthesized attributes, and the inherited attributes of its children exactly once (with a correct type). Attribute references in the expressions may refer to the inherited attributes of the production, or the synthesized attributes of the children. There is one exception: the inherited attributes of the merged child (e.g. *res*) may not be defined, and the synthesized attributes of the merging children (e.g. *left* and *right*) may not be referred to.

We define a translation to Haskell (denotational semantics) that gives both a static and operational semantics to SAGs. If the generated Haskell program is type correct then so is the *sagcore* program. The execution of the generated Haskell functions shows how the rules are used to construct the attributed trees.

As mentioned in Section 7.3, we translate a semantics-block to an execution plan of the production. We use a naming conventional to translate AG names to Haskell names. Attributes are referred to by an identifier *cXa*. In this notation, *c* and *a* are the name of the child and the name of the attribute respectively. *X* is a subscript *I* for an inherited attribute, and *S* for a synthesized attribute. Merging children are prefixed with an underscore (we assume that names of children do not start with an underscore).

Each rule corresponds to an instruction in the execution plan. In the end, we return a tuple with values for the synthesized attributes:

$$\begin{array}{lll} [\![\textbf{sem}\ I\ \overline{r}]\!] & & \rightsquigarrow \lambda\, lhs_I ins_1 \ldots lhs_I ins_n \rightarrow \textbf{do rec} \\ & & \qquad \{ [\![\overline{r}]\!]; return\ (lhs_S out_1, \ldots, lhs_S out_m) \} \\ [\![\textbf{child}\ c :: I = e]\!], & \text{conventional} & \rightsquigarrow (c_S out_1, \ldots, c_S out_m) \leftarrow e\ c_I ins_1 \ldots c_I ins_n \\ [\![\textbf{child}\ c :: I = e]\!], & \text{merged} & \rightsquigarrow \textbf{let}\ \_c = e\ c_I ins_1 \ldots c_I ins_n \\ [\![p = e]\!] & & \rightsquigarrow \textbf{let}\ [\![p]\!] = [\![e]\!] \\ [\![\textbf{merge}\ cs\ \textbf{as}\ c : N = e]\!] & & \rightsquigarrow (c_S out_1, \ldots, c_S out_m) \leftarrow [\![e]\!]\ \_c_1 \ldots \_c_k \end{array}$$

The indices *m* and *n* range over the inherited and synthesized attributes of nonterminal *I*.

In previous work [Middelkoop et al., 2010a], we described different (more sophisticated) translations of (Ordered) Attribute Grammars to execution plans (in Haskell). The merge-syntax is fully compatible with those translations.

7.C Semantic Lookahead

In this section, we show how to deal with a choice that has a potential global effect on attributes of the tree. For example, suppose that we deal with a type inferencer that needs to choose between *int* and *double* for the type of a numerical constant. This choice may have a global effect: a wrong choice potentially causes a typing error in the remainder of the program to type. To explore such a choice, we want to look at the steps of a child *and* the steps that the remaining computation gives if we would choose that child. In terms of the monad: if k is the continuation after the choice, i.e. $\mathit{choose}\ l\ r \gg\!= k$, then we want to lift the choice to $\mathit{choose}\ (l \gg\!= k)\ (r \gg\!= k)$.

We provide a monadic operation *Ahead* (explained below) to access the continuation k. This operations comes at a price: since the continuation is not known until runtime, so we wish our choose-function to work for arbitrary continuations. In particular, we refrain from making static assumptions on the type of the result the continuation computes.

```
data Stepwise i a where
  Ahead :: (forall b.(a → Stepwise i b) → Stepwise i b) → Stepwise i a
```

Ahead takes as argument a function f that takes the continuation k as argument. To use *Ahead*, we provide this function f. The continuation takes the result that we are supposed to compute, and returns a computation of some existential type b. The computation passed to ahead thus wraps around the computation: it specifies an input for the continuation, and allows us to modify the result of the continuation. Thus, this lifted the choice to toplevel, as mentioned in the previous paragraph. Also, modifying the result of the continuation instead of building an input for the continuation based on trying out the continuation is what makes this approach different from Continuation Passing Style's *call/cc*.

As an example, a computation m is equivalent to $\mathit{Ahead}\ (\lambda k \rightarrow m \gg\!= k)$. As another example, a global choice can be made using:

$$\mathit{Ahead}\ (\lambda k \rightarrow \mathit{choose}\ (l \gg\!= k)\ (r \gg\!= k))$$

Although we cannot make an assumption about the type of the result of k, we can make an assumption on the type of progress reports, and thus use the contents of the progress reports to direct the exploration between the two choices (depending on the search strategy).

When we encounter an $\mathit{Ahead}\ f$ in *lazyEval*, the question is what continuation we pass in. The function f is required to get the computation that resembles the remaining continuation. However, since we are in *lazyEval*, the continuation cannot return any progress reports, and cannot observably fail. Thus we simply pass in *Return*, which succeeds immediately, and gives us the result that f passes to its continuation:

$$\mathit{lazyEval}\ (\mathit{Ahead}\ f) = f\ \mathit{Return}$$

When we encounter an *Ahead f* in *smallStep* without parents on the parent stack, we pause the computation. The evaluation can only continue if the caller specifies how the computation proceeds (possibly by calling *Ahead* itself):

$$\begin{array}{l}\textbf{data}\ \mathit{Report\ i\ a}\ \textbf{where}\\ \quad \mathit{Lookahead} :: (\mathit{forall}\ b.(a \to \mathit{Stepwise\ i\ b}) \to \mathit{Stepwise\ i\ b}) \to \mathit{Report\ i\ a}\\ \mathit{smallStep}\ (\mathit{Ahead}\ f) = \mathit{Lookahead}\ f\end{array}$$

If there are parents on the parent stack, however, we continue to reduce f. The continuation to pass to f are the remaining parents on the parent stack:

$$\mathit{reduce}\ (\mathit{Ahead}\ f)\ (\mathit{Bind}\ g\ r) = \mathit{smallStep}\ \$ f\ \$\ \lambda a \to \mathit{Pending}\ (g\ a)\ r$$

The use of *Ahead* has an interesting interplay with the hybrid evaluation model. On-demand evaluation skips progress reports, and passes a *Return* as continuation. If a parent of a child that uses lookahead is evaluated on-demand, then the lookahead of the child does not observe the skipped progress reports. So, lookahead does not see beyond on-demand evaluated AST nodes. So, if a progress report contains information essential to a choice using lookahead, we need to take sufficient *smallStep*s at a common ancestor node such that the lookahead observed the report. For example, we can emit a progress report e.g. *DoneGreedy* when a choice using lookahead inspected the progress reports it was interested in, take *smallStep*s at the root of the tree (or a common ancestor), until we encounter *DoneGreedy*, then switch to *lazyEval*.

Also, *Ahead* has an interplay with multi-visit AGs (deriveable from Ordered Attribute Grammars). Without *Ahead*, stepwise evaluation yields a *Done* for a child when the *first* visit is finished. When using *Ahead*, however, yields a *Done* when the continuation finished with its *last* visit. Again, the progress report mechanism can be used to limit the exploration to certain visits, or yield the outcome of a visit as intermediate result.

It is advisable to ensure that all choices can be made based on results of the first visit. If it requires a progress report that is emitted in a second visit, this requires the first visit to finish completely, which is likely already a full exploration of the tree (for that choice). It is possible to make the approach more flexible and offer just on-demand evaluation for first visits, and (hybrid) stepwise evaluation for later visits. That would allow exploration in a later visit based on (lazily) computed results in earlier visits. This requires visits to be made explicit in the AG specification (Chapter 3 and Chapter 5).

7.D Watchers

With the approach described so far, the type i of a progress is fixed: given an $m \ggg k$, both m and the computation returned by k must have the same i type. This limitation affects the compositionality of stepwise computations. Also, sometimes we wish to return progress reports of the same type as the result of the computation[12]. On the other hand, since the type

[12] See Example 7 of `Examples.hs` in the cabal package at `https://svn.science.uu.nl/repos/project.ruler.papers/archive/stepwise-1.0.2.tar.gz`

i is fixed, we know that a continuation has this type, thus when using *Ahead* (Appendix 7.C) we can inspect the progress reports of the continuation. This we cannot do without knowing *i*.

To get the best of both ways, we parametrize the type *i* with a type *w* that functions as an index, the *watcher*. A computation has the type *Stepwise i w a*, which returns progress reports with values of the type *i w*. When *w* is an existential type, e.g. when using *Ahead*, we can still scrutinize on all values not dependent on *w*. If *w* is a concrete type, we can scrutinize the depending values as well.

To embed a computation with a different watcher type, we provide a transcoding operation:

```
data Stepwise i w a where
  Transcode :: (i v → [i w]) → Stepwise i v a → Stepwise i w a
```

It takes a progress report of type *i v* and converts it to zero or more progress reports of type *i w*. For example, *smallStep* (*Transcode* (*const* [ ]) *m*) does not return any progress reports. In the actual implementation, we maintain composed transcoders on the parent stack, such that we can immediately apply them without having to traverse the stack.

In practice, we also allow the transcoding function to store a local state, such that it can remember a number of progress reports and combine them into a single progress report (compression). If paths in the tree are long, and many nodes are inspecting and passing on progress reports, then each node gets many reports to process (especially near the root). With the transcoding mechanism, we can trade evaluation granularity for the number of reports.

7.E Additional Tree Structure

In the example in Section 7.2, the children to choose from are part of the original AST, and reduced my the merge-rule to a single virtual child. This is not always the case, as the example in this section shows.

Suppose that for a simply-typed lambda calculus, we want to specify two alternative productions for an application-expression: one alternative represents strict application, the other lazy application. As an optimization, we may choose a strict application when all functions that can occur as left-hand side are strict in their respective argument. To encode this choice between productions, we need to encode these choices as additional children that are not present in the original AST. One option is to transform the tree on-demand, prior to attribute evaluation. As alternative and more elegant approach, we can use a Higher-order Attribute Grammar and use nonterminal attributes (or: *higher-order attributes*). In this section, we demonstrate how.

We sketch a design pattern for multiple alternative productions by example, using several UUAG features. The goal is to be able to define multiple semantics for a production, without affecting the original context-free grammar. We start with the context-free grammar of the example:

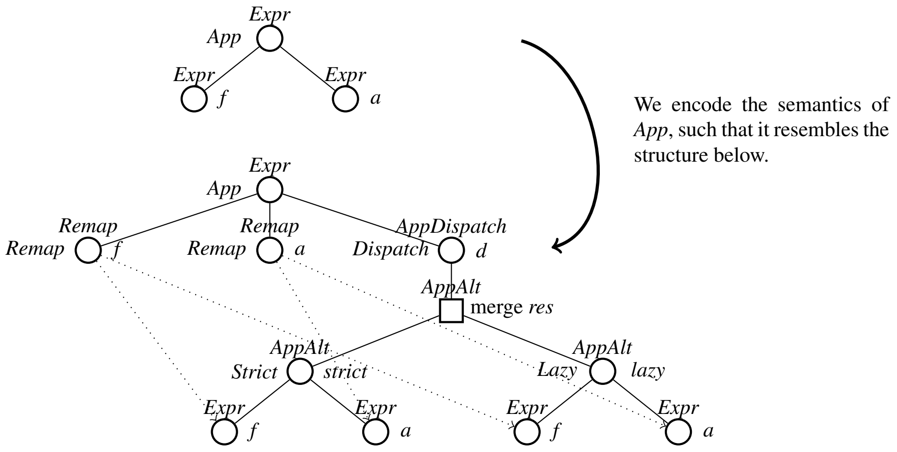
```
data Expr
  | ...                          -- Some lambda calculus.
  | App   f :: Expr   a :: Expr  -- Conventional application.
```

We concern ourselves only with the *App* production, which has two children, *f* and *a* respectively.

We introduce additional nonterminals to represent the choices: a nonterminal *AppDispatch* with production *Dispatch* that has a child for each alternative production. In particular, the semantics of *Dispatch* deals with the choice—*dispatch*—between the alternatives. Furthermore, we introduce a nonterminal *AppAlt*, containing the alternative productions. Their structure is a clone of the *App* production. Being separate productions, we can give each its own distict semantics:

```
data AppDispatch | Dispatch    -- Introduces two children of nonterm AppAlt.
  strict :: AppAlt    lazy :: AppAlt
data AppAlt                    -- Productions for the above children.
  | Strict     f :: Expr   a :: Expr   -- Intended for child strict.
  | Lazy       f :: Expr   a :: Expr   -- Intended for child lazy.
sem AppDispatch | Dispatch     -- Merges the children.
  merge strict lazy as res :: AppAlt = choose_app
sem AppAlt ...                 -- give some semantics to Strict and Lazy.
```

With these additional nonterminals and productions, we encode the semantics of *App*, such that it literally encodes the choices as additional tree nodes. Below, we sketch the original AST structure for *App*, and the intended structure. The nodes are displayed as a circle. The nonterminal corresponding to a node is displayed above it, the production to the left, and the name to the right. The merge is displayed as a square. The dotted arrows represent a transfer of the semantics of a node to a different location in the tree.



To establish the the intended structure, we need to accomplish two tasks. Since we deal with the actual semantics of *App* in the productions *Strict* and *Lazy*, it is actually inconvenient to have *f* and *a* as children of *App*: we wish to take these children away, and attach them instead below the *strict* and *lazy* nodes. Furthermore, we wish to construct a child *d* that dispatches to *strict* or *lazy*.

For the first task, we *replace* the semantics of f and a (nonterminal *Expr*) with a nonterminal *Remap I_Expr* that provides the original semantics as a synthesized attribute, and is described by:

```
data Remap a | Remap   s :: a        -- Terminal s stores semantics of type a.
attr Remap   syn s :: a              -- Attribute s provides it to parent.
sem Remap | Remap   lhs.s = s        -- Rule for s
```

To replace f and a's semantics, we declare two nonterminal attributes f and a for *App* that clash with the names of the original children f and a. The value for the nonterminal attribute must be a *semantics transformer*: a function that takes the original semantics (*I_Expr*) and provides the transformed semantics (*Remap I_Expr*), denoted as follows[13]:

```
sem Expr | App
  child f :: Remap = λorigSem → sem_Remap origSem   -- Transforms f.
  child a :: Remap = λorigSem → sem_Remap origSem   -- Transforms a.
```

The original semantics of f and a is passed as argument, thus we use it to store it as terminal in the *Remap* node. Since f and a effectively now are of nonterminal *Remap*, it means we can use attributes $f.s$ and $a.s$ to obtain the original semantics of f and a and use it to construct child d.

For the second task, we construct d, again using a nonterminal attribute[14]:

```
sem Expr | App
  child d :: AppDispatch = Dispatch (Strict f.s a.s) (Lazy f.s a.s)
```

The right-hand side is a proper semantics for *AppDispatch*, and describes the tree for d as we visualized above.

Via these nonterminal attributes, we managed to tweak the tree structure, such that it conveniently encodes the choices that need to be made. The additional nodes still lack attributes: they need at least the same attributes of *Expr*, and perhaps more if we want to. To prevent having to write the attributes of *Expr* twice, we use UUAG's nonterminal sets to define for example an inherited environment attribute:

```
set AllExpr = Expr AppDispatch AppAlt   -- Defines a set of nonterminals
attr AllExpr   inh env :: Env           -- Declares env for the three nonterms.
```

[13] UUAG's actual syntax for higher-order attributes differs slightly:

```
sem Expr | App
  inst.f :: Remap {T_Expr}                    -- type signature
  inst.f = λorigSem → sem_Remap origSem       -- conventional rule
```

[14] For nonterminal attribute d, we do not get the original semantics of d as parameter by absence of a conventional child d. The syntax for a nonterminal attribute thus adds or replaces depending on the existence of a conventional child. Also, there may not be duplicate nonterminal attributes to prevent ordering issues.

There is no need to define rules for the attributes of *AppDispatch* and *App*: these are automatically provided by the copy-rule mechanism. Inherited and synthesized are respectively passed topdown or bottom up when no explicit rules are specified.

In summary, we can easily encode a choice between productions as a choice between children. In particular, using UUAG's higher-order extensions and copy rules, this transformation has a very concise and orthogonal implementation.

7.F Inference Rules and AGs

In this section we show an example that is closer related to type systems compared to Section 7.2. Consequently, it is also more complicated. The example is an equality inferencer, implemented with AGs. Given two objects (o_1 and o_2 respectively), and a set of equality assumptions Γ, we wish to obtain a derivation $\Gamma \vdash o_1 \equiv o_2$ using the following conventional reflection, symmetry and transitivity inference rules:

$$\Gamma \vdash o \equiv o \ \textsc{refl} \qquad \frac{\Gamma \vdash o_2 \equiv o_1}{\Gamma \vdash o_1 \equiv o_2}\ \textsc{sym} \qquad \frac{\begin{array}{c}\Gamma \vdash o_1 \equiv o_2 \\ \Gamma \vdash o_2 \equiv o_3\end{array}}{\Gamma \vdash o_1 \equiv o_3}\ \textsc{trans} \qquad \frac{(o_1, o_2) \in \Gamma}{\Gamma \vdash o_1 \equiv o_2}\ \textsc{assum}$$

Clearly, these rules are not syntax directed: a derivation with these rules depends on both objects. The rules are ambiguous. For example, we can apply SYM twice and end up where we started. To disambiguate, we search for a derivation with minimal depth, in a fixed but unimportant order. Also, in rule TRANS, o_2 is not known a priori and needs to be guessed. To simplify the example, we assume that the objects are finitely orderable, such that simply try values for o_2, instead of having to resort to variables and unification.

We start with a grammar E for equality derivations, and attributes that represent the parameters of the equality relation. As output, we define an attribute *pp* that is a string representation of the derivation:

```
data E | Any | Refl | Sym | Trans | Assum
attr E   inh l, r :: Obj env :: Env   syn pp :: String
type Env = Set (Obj, Obj)
```

The productions are all empty. We define per production nonterminals for each premise, using higher-order attributes. For example, the production *Any* makes a choice between any of the other productions:

```
sem E | Any
  child refl   : E = sem_Refl
  child sym    : E = sem_Sym
  child trans  : E = sem_Trans
  child assum  : E = sem_Assum
  merge refl sym trans assum as prf : E = λe1 e2 e3 e4 →
    memo (lhs.l, lhs.r) $ info Work ≫ one localChoice [e1, e2, e3, e4]
  lhs.pp = prf.pp
one = foldl f (fail "")
```

A child-rule introduces a nonterminal where the semantics is defined by the RHS of the child rule (Appendix 7.E). We assume that if a rule for an inherited attribute of a child is omitted, the value is taken from an equally named attribute of *lhs* instead. Note that *prf* does not have any inherited attributes, because it's a merged child.

One of the four alternative children is selected by *one*. It takes the first child to succeed, or the last one to fail (left biased) and calls it *prf*. By adding a progress report in front of the selected choice, the number of progress reports stands for the depth of the derivation. Later we see a variant of *localChoice*, which can look ahead beyond the child to see if evaluation after we pick a certain candidate succeeds.

Instead of a derivation tree, we build a derivation graph via memorization. It remembers the outcome using as key $(lhs.l, lhs.r)$. If it encounters the same key while actually defining itself, it causes a backtrack (prevents cycles). Also, lookaheads cannot see beyond *memo*.

To implement SYM, we search for a derivation with the inputs swapped:

```
sem E | Sym
  child prf : E = sem_Any
  prf.l  = lhs.r
  prf.r  = lhs.l
  lhs.pp = "sym" ++ prf.pp
```

To implement REFL, we need to check that the left and right sides are equal. Similarly, for ASSUM, we check that the equality occurs as assumption in the environment. For that, we introduce an additional nonterminal, *C*. It takes a Boolean value as inherited attribute. If this value is *True*, if defines the *pp* attribute, otherwise it fails the evaluation. We encode this behavior using a merge with an empty set of children:

```
data C | Check
attr C   inh guard :: Bool   syn pp :: Doc
sem C | Check
  merge as res : C = if lhs.guard then pp "check" else fail "guard"
  lhs.pp = res.pp
sem E | Refl
  child c : C = sem_Check
  c.guard = lhs.l ≡ lhs.r
  lhs.pp  = "refl" ++ c.pp
sem E | Assum
  child c : C = sem_Check
  c.guard = (lhs.l, lhs.r) ∈ lhs.env
  lhs.pp  = "assum" ++ c.pp
```

Finally, TRANS. We use a nonterminal *G* to guess the intermediate term. For that, we enumerate objects, and select one of them, using *lookahead* via *globalChoice*. This lookahead is essential, because we need to pick an intermediate term that works with both premises of TRANS:

```
data G | Guess
attr G   syn obj :: Obj
sem G | Guess
  merge as res : G = one globalChoice $ map return enumerate
  lhs.obj = res.obj
sem E | Trans
  child g    : G = sem_Guess
  child prf1 : E = sem_Any
  child prf2 : E = sem_Any
  prf1.l = lhs.l
  prf1.r = g.obj
  prf2.l = g.obj
  prf2.r = lhs.r
  lhs.pp = "trans" ++ prf1.pp ++ prf2.pp
```

The inference rules look rather innocent, but an inferencer for them is far from trivial. With our AG extension, we can now relatively easily deal with such inference rules.

7.G Mutual Expressibility of Declarative Aspects in Rules and Productions

We hinted in Section 7.1 that declarative aspects in rules and productions are mutually expressible, and can both be encoded as a choice between children. We showed the latter for productions in Appendix 7.E. For declarative aspects in rules we actually need to qualify our statement: it works for finite and inductively defined data types[15]. We give an example in this section.

Suppose that types are inductively defined as follows:

```
data Ty = Int | Arrow Ty Ty
```

In an AG for a type inferencer, we wish to define a semantics for the lambda production that infers the type of the argument. When we assume that we can obtain somehow a nondeterministic value *fresh*, we could write the semantics for a lambda expression as follows:

```
attr Expr   inh env :: [(String, Ty)]   syn ty :: Ty
sem Expr | Lam
  loc.argTy = fresh
```

[15] Technically speaking, the statement holds for any value represented by a finite number of bits (which is the case for any value in practice).

```
    body.env = x : loc.argTy, lhs.env
    lhs.ty   = Arrow loc.argTy body.ty
```

We nondeterministically get a type for the argument, and use this in the environment for the body of the lambda, and for its type. Note that the definition for *loc.argTy* clashes with referential transparency: *fresh* is not associated with a single unique value. Also note that there may be infinitely many values possible for *argTy*: for predictability reasons, we want to make explicit what value is chosen here.

We can encode the above declarative aspect in a functional way: we define a tree that represents all possible values of type *Ty*, and use the choice function *globalChoice* (Appendix 7.F) to choose to smallest *Ty* that results in a successful inference.

First, we introduce a nonterminal attribute of nonterminal *FreshDispatch* (defined later) that provides us with a synthesized attribute *ty* that holds the fresh type:

```
sem Expr | Lam
  child fr :: FreshDispatch = mkFresh
  loc.argTy = fr.ty
```

Then, *mkFresh* builds such an infinite tree of fresh possibilities for a type. We follow the pattern described in Appendix 7.E: *FreshDispatch* corresponds to a dispatcher node that has a child (respectively, *int* and *arrow*) for each alternative type:

```
mkFresh = FreshDispatch FreshInt (FreshArrow mkFresh mkFresh)
data FreshDispatch | Dispatch   int :: FreshAlt   arrow :: FreshAlt
data FreshAlt
  | FreshInt
  | FreshArrow   arg :: FreshDispatch   res :: FreshDispatch
attr FreshDispatch FreshAlt   syn ty :: Ty
```

The semantics of *Dispatch* defines that (with a preference for *int*) the child that makes the evaluation succeed (for a notion of success defined by the other code of the inferencer) is chosen:

```
sem FreshDispatch | Dispatch
  merge int arrow as res = globalChoice
sem FreshAlt
  | FreshInt     lhs.ty = Int
  | FreshArrow lhs.ty = Arrow arg.ty res.ty
```

These declarative forms are thus indeed mutually expressible. In practice, however, we typically deal with declaratively defined attributes in a different way: we use a unification-based approach together with an additional substitution attribute, which does not need an exploration of alternatives. While we can encode the declarative aspect via such an exploration, there may be more efficient ways to resolve it.

7.H Proofs

In this section, we present proofs of some important properties of the stepwise monad. For simplicity, we take the implementation of Section 7.4, without the extensions mentioned in Section 7.6. We prove that the monad laws hold. This is important, because the desugaring of monad notation makes use of them. Also, it is a sanity check that gives additional confidence in the correctness of the implementation. We need to prove that:

Left identity: $\mathit{return}\ a \gg\!= f \equiv f\ a$
Right identity: $m \gg\!= \mathit{return} \equiv m$
Associativity: $(m \gg\!= f) \gg\!= g \equiv m \gg\!= (\lambda x \to f\ x \gg\!= g)$

In our case, we have to prove the monad laws twice, because we have two interpretations of the monad: *lazyEval* and *smallStep*.

7.H.1 Monad Laws for Lazy Evaluation

These proofs are straightforward equivalence proofs that exploit Haskell's referential transparency to reduce the LHS and RHS to a common term:

$$\begin{aligned}
&\mathit{lazyEval}\ (\mathit{return}\ a \gg\!= k)\\
&\quad\equiv \mathit{lazyEval}\ (\mathit{Pending}\ (\mathit{Return}\ a)\ (\mathit{Bind}\ k\ \mathit{None}))\\
&\quad\equiv \mathit{evalPending}\ (\mathit{Bind}\ k\ \mathit{None})\ (\mathit{lazyEval}\ (\mathit{Return}\ a))\\
&\quad\equiv \mathit{evalPending}\ \mathit{None}\ (\mathit{lazyEval}\ (k\ a))\\
&\quad\equiv \mathit{lazyEval}\ (k\ a)
\end{aligned}$$

$$\begin{aligned}
&\mathit{lazyEval}\ (m \gg\!= \mathit{return})\\
&\quad\equiv \mathit{lazyEval}\ (\mathit{Pending}\ m\ (\mathit{Bind}\ \mathit{Return}\ \mathit{None}))\\
&\quad\equiv \mathit{evalPending}\ (\mathit{Bind}\ \mathit{Return}\ \mathit{None})\ (\mathit{lazyEval}\ m)\\
&\quad\equiv \mathit{evalPending}\ \mathit{None}\ \$\ \mathit{lazyEval}\ \$\ \mathit{Return}\ \$\ \mathit{lazyEval}\ m\\
&\quad\equiv \mathit{lazyEval}\ m
\end{aligned}$$

In case of the associativity law, we reduce both sides to a chain of *lazyEval* calls. This is the intuition behind *evalPending*: for each parent on the pending stack, it passes to the parent the *lazyEval* of its child:

$$\begin{aligned}
&\mathit{lazyEval}\ (m \gg\!= (\lambda x \to k\ x \gg\!= h))\\
&\quad\equiv \mathit{lazyEval}\ (\mathit{Pending}\ m\ (\mathit{Bind}\ (\lambda x \to \mathit{Pending}\ (k\ x)\ (\mathit{Bind}\ h\ \mathit{None}))\ \mathit{None}))\\
&\quad\equiv \mathit{evalPending}\ (\mathit{Bind}\ (\lambda x \to \mathit{Pending}\ (k\ x)\ (\mathit{Bind}\ h\ \mathit{None}))\ \mathit{None})\ (\mathit{lazyEval}\ m)\\
&\quad\equiv \mathit{lazyEval}\ \$\ (\lambda x \to \mathit{Pending}\ (k\ x)\ (\mathit{Bind}\ h\ \mathit{None}))\ \$\ \mathit{lazyEval}\ m\\
&\quad\equiv \mathit{lazyEval}\ \$\ \mathit{Pending}\ (k\ (\mathit{lazyEval}\ m))\ (\mathit{Bind}\ h\ \mathit{None})\\
&\quad\equiv \mathit{lazyEval}\ \$\ h\ \$\ \mathit{lazyEval}\ \$\ k\ \$\ \mathit{lazyEval}\ m\\
&\quad\equiv \mathit{lazyEval}\ \$\ h\ \$\ \mathit{evalPending}\ (\mathit{Bind}\ k\ \mathit{None})\ \$\ \mathit{lazyEval}\ m\\
&\quad\equiv \mathit{lazyEval}\ \$\ h\ \$\ \mathit{lazyEval}\ (\mathit{Pending}\ m\ (\mathit{Bind}\ k\ \mathit{None}))\\
&\quad\equiv \mathit{evalPending}\ (\mathit{Bind}\ h\ \mathit{None})\ (\mathit{lazyEval}\ (\mathit{Pending}\ m\ (\mathit{Bind}\ k\ \mathit{None})))\\
&\quad\equiv \mathit{lazyEval}\ (\mathit{Pending}\ (\mathit{Pending}\ m\ (\mathit{Bind}\ k\ \mathit{None}))\ (\mathit{Bind}\ h\ \mathit{None}))\\
&\quad\equiv \mathit{lazyEval}\ ((m \gg\!= k) \gg\!= h)
\end{aligned}$$

7.H.2 Monad Laws for Stepwise Evaluation

The proof for left-identity is still a straightforward equivalence proof:

```
smallStep (return a >>= k)
  ≡ smallStep (Pending (Return a) (Bind k None))
  ≡ reduce (Return a) (Bind k None)
  ≡ reduce (k a) None
  ≡ smallStep (k a)
```

The remaining proofs are more complicated. We quickly run into the problem that we cannot rewrite any further because we do not know some of the variables, e.g. *m*:

```
smallStep (m >>= return)
  ≡ reduce m (Bind Return None)
  ...
  ≡ smallStep m
```

To proceed, we use structural induction and case distinction on all possible values of *m*, which means that we use induction over the sequence of progress reports yielded by *m*. First the inductive case:

```
case m of
  Yield i n → reduce (Yield i n) (Bind Return None)
              ≡ Step i (Pending n (Bind Return None))
              ≡ Step i (n >>= return)    -- induction hypothesis
              ≡ Step i n
              ≡ smallStep (Yield i n)
```

In contrast to many inductive proofs, actually the base case is the interesting part. We finish off most cases via trivial equality rewrites. Unfortunately, when *m* is a *Pending*-value, we are again stuck:

```
case m of
  Yield i n     → ⊥    -- Falsum: no progress report
  Fail s        → reduce (Fail s) (Bind Return None)
                  ≡ Failed s
                  ≡ smallStep (Fail s)
  Return v      → reduce (Return v) (Bind Return None)
                  ≡ reduce (Return v) None
                  ≡ smallStep (Return v)
  Pending n p   → reduce (Pending n p) (Bind Return None)
                  ≡ reduce n (push p (Bind Return None))
                  ...
                  ≡ reduce n p
                  ≡ smallStep (Pending n p)
```

We proceed with structural induction on *n* and *p*, such that we may continue to unfold *reduce*. The cases for *Yield*, *Fail* and *Pending* for *n* are similar to cases above. Of interest are the two cases with *None* and *Bind* for *p*. We may apply this induction hypothesis, as well as the earlier one above:

```
case p of
  None →     reduce n (push None (Bind Return None))
               ≡ reduce n (Bind Return None)   -- I.H (above)
               ≡ smallStep n
               ≡ reduce n None
  Bind f q → reduce n (push (Bind f q) (Bind Return None))
               ≡ reduce n (Bind f (push q (Bind Return None)))   -- I.H.
               ≡ reduce n (Bind f q)
```

The proof for the associativity law proceeds in a similar way.

7.I Benchmarks

Regarding asymptotic complexity, it is easy to show that in the worst case the performance is exponential in the number of choice points. For small inputs, this may not be a problem, and in general it is up to the programmer to keep alternatives to a limit. A remaining question, however, is how much overhead the stepwise evaluation induces. This overhead should be constant, and we experimentally validated that this is indeed the case.

For this purpose, we compared the evaluation of a sequences of binds, using both the stepwise monad and the identity monad[16]. Benchmark `id` uses the identity monad, `lazy` uses *lazyEval*, and `step` uses full stepwise evaluation. We tested both nested binds, and long sequences of binds, via the following benchmark:

```
bindbench length depth = tree depth where
  runner 0 n  = n `seq` return ()
  runner m n = tree n >>= λ() → runner (m − 1) n
  tree 0 = return ()
  tree n = runner length (n − 1)
```

We ran the benchmarks on a standard MacBook 2.1 (2 GB RAM, 2 GHz Core 2 Duo) with GHC 6.12.1 and GCC 4.2.1. The outcome was measured via the `criterion` package (version 0.5.05). It executed the benchmarks hundreds of times. We used `+RTS -H512M` as commandline parameter to minimize the effect of garbage collection. The variance of the measured results was less than a percent.

[16] See `https://svn.science.uu.nl/repos/project.ruler.papers/archive/StepBenchmarks.hs`.

depth	length	id (ms)	lazy (ms)	step (ms)
1	10,000	0.0555	1.40	1.42
1	100,000	0.554	20.2	14.4
1	1,000,000	5.56	259	144
5	2	0.000540	0.0102	0.0137
7	2	0.00230	0.0415	0.0564
9	2	0.00905	0.166	0.231
5	3	0.00214	0.0585	0.0809
3	50	0.712	18.2	26.0
4	10	0.0680	1.63	2.35

Indeed, both the evaluation via the identity monad and full stepwise evaluation show a constant amount of overhead per bind, with about a factor 30 throughput difference. Lazy evaluation shows a more erratic behavior. With an earlier and simpler version of our library – essentially the implementation of Section 7.4 – we measured a factor 10 throughput difference. The additional complexity discussed in Section 7.6 thus has its cost. However, this benchmark does not show us what to expect in a real application.

To measure actual performance in practice, we compiled UHC (repository version 2226, 04 dec 2010) with both the conventional AG translation (using the `--optimize` flag) and the stepwise AG translation. For the latter, we added the `Control.Monad.Stepwise.AG` module to the export list of UHC's `Common.chs`. Furthermore, we added the commandline option `--with-uuagc-options="--breadthfirst"` to the call to UHC's `configure`. Since the AGs of UHC are orderable, we can fully evaluate them stepwise, without requiring *lazyEval*. Three insignificant AGs were not orderable: for these we used the conventional translation. We believe that the UHC is a good benchmark to validate AG performance in practice: it uses hundreds of AGs for its implementation, such as AGs that work on large data structures (the ASTs of various Haskell programs), but also on thousands on small data structures (for example, to compute the free variables of a type). The AGs also vary on the number of productions, and the number of rules.

We measured the time that it took for UHC to compile programs from the `nofib` benchmark suite. Additionally, we verified that the outcome and performance of the compiled Haskell programs are not affected by the difference in implementation of UHC. The outcome of these benchmarks, combined with the proofs of Section 7.H and the strong typing discipline imposed on the monad, gives us confidence in the correctness of the implementation.

benchmark (name)	conventional (sec.)	stepwise (sec.)
`imag/bernouilli`	4.51	4.95
`imag/binarytrees`	4.35	4.40
`imag/digits-of-e1`	4.14	4.16
`imag/digits-of-e2`	3.85	3.88
`imag/exp3_8`	4.33	4.29
`imag/gen-regexps`	4.26	4.38
`imag/integrate`	4.29	4.34
`imag/loop`	3.76	3.81
`imag/nsieve`	1.11	1.10
`imag/paraffins`	5.55	5.62
`imag/partial-sums`	4.40	4.48
`imag/pidigits`	4.60	4.29
`imag/primes`	3.77	3.72
`imag/queens`	3.74	3.81
`imag/recursive`	4.37	4.37
`real/infer`	11.9	11.8

The difference in performance is very small. On average, the performance using stepwise computations is marginally worse. However, with such small differences, the garbage collector has more impact on performance, as the cases for `imag/nsieve` and `real/infer` seem to suggest. With these results, we dare to say that our implementation is ready to be used in practice.

7.J Java Implementation

In Section 7.5, we gave a short overview of an imperative implementation of the work as described in this chapter. In this section, we work out the example of Section 7.2 using Java instead of Haskell. We derive the code for the example in a systematic way. We thus describe implicitly how to translate AGs to Java. In comparison to AG systems such as JastAdd [Ekman and Hedin, 2007], our approach offers lazy and strict evaluation, higher-order attributes, and the stepwise features of this chapter. Furthermore, there is a clear correspondence between the Java implementation and the reference implementation in Haskell.

As conventional in object-oriented languages with class-based inheritance, we introduce an abstract class for each nonterminal and a subclass for each production. Nodes of the AST are instances of these subclasses. A child of a production has the abstract class as its static type, and one of the subclasses as its runtime type. The abstract class contains the inherited and synthesized attributes of the nonterminal as fields. A subclass contains the rules of the production (and possibly local attributes). For the example, we thus introduce a abstract class *Pred*, and a subclass for each production:

```
public final class PredConst extends Pred    /* implemented later */
public final class PredVar   extends Pred    /* implemented later */
public final class PredLet   extends Pred    /* implemented later */
public final class PredOr    extends Pred    /* implemented later */
public final class PredAnd   extends Pred    /* implemented later */
```

The constructors of the subclasses take values for the symbols of the associated production as parameter. For example, we can construct the predicate p_3 that represents '*x* or *false*':

```
Pred p1 = new PredVar("x");
Pred p2 = new PredConst(false);
Pred p3 = new PredOr(p1, p2);
```

We show later how to pass values for inherited attributes to such a node, and how to extract values for synthesized attributes. We first show how attributes, rules and attribute evaluation are represented.

The abstract class *Pred* contains inherited and synthesized attributes in the respective fields *_inhs* of type *PredInh* and *_syns* of type *PredSyn*. The inherited attributes are publicly accessible via the getter method *inhs*. The synthesized attributes may only be accessed directly by the node itself. We show later that a parent node uses the method *lazyEval* on a child (inherited from the class *Node*) to access the child's synthesized attributes. The *Node* type is parameterized with tree types: the type of the object containing the synthesized attributes, the type of the information messages that may be yielded during strict evaluation, and the types of failures it may generate.

```
public abstract class Pred extends Node<PredSyn, Info, BacktrackException> {
  private PredInh _inhs;
  private PredSyn _syns;

  public Pred() {
    _inhs = new PredInh();
    _syns = new PredSyn(); }

  protected PredSyn syns() {
    return _syns; }

  public PredInh inhs() {
    return _inhs; }}

public class PredInh {
  private Attr<HashMap<String, Boolean>> _env;

  public PredInh() {
    _env = new Attr<HashMap<String, Boolean>>(); }

  public Attr<HashMap<String, Boolean>> env() {
    return _env; }}

public class PredSyn {
  private Attr<Boolean> _value;

  public PredSyn() {
    _value = new Attr<Boolean>(); }

  public Attr<Boolean> value() {
    return _value; }}
```

Attributes are parameterized by the type of values that they store. Initially, an attribute is initialized to *null*. An attribute can be assigned a value in two ways. Either a value can be explicitly set via the attribute's **set** method, or a rule can be associated via the method *dependsOn*. In the latter case, when the value of the attribute is requested via the *get* method and the attribute is not yet defined, the associated rule is run. The rule calls the **set** method to assign a value to the attribute. Therefore, we require rules to be encoded as objects, such that they can be passed around as first-class citizens.

```
public final class Attr<T> {
  private T _value;
  private Runnable _rule;

  public Attr() {
    _value = null;
    _rule = null; }

  public T get() {
    if (_value == null && _rule != null)
      _rule.run();

    return _value; }

  public void set(final T value) {
    _value = value; }

  public void dependsOn(final Runnable rule) {
    _rule = rule; }}
```

The *Node* class from which all AST nodes inherit exposes the synthesized attributes only to subclasses of *Node*. The intended usage protocol is that after creation of a child node, its parent first assigns rules to the node's inherited attributes. Subsequently, the parent may issue a call to the *begin* method to obtain a stepwise computation. A stepwise computation of a child represents the computations for the subtree rooted by that child (captured by the class *Parents* that we discuss later). A stepwise computation can be manipulated in two ways. It can be lazily evaluated via *lazyEval*, such that it returns the node's synthesized attributes, or it can perform one evaluation step, and return a progress report (the class *Report* that we discuss later).

```
public abstract class Node<X, I, E> extends CoroutineBase<I, E> {
  public Node() {}

  abstract protected X syns();

  public Stepwise<X, I, E> begin() {
    return new Parents<X, I, E>(this, syns()); }}

public abstract class CoroutineBase<I, E> implements Coroutine<I, E> {
  /* implemented later */ }

public interface Stepwise<X,I,E> extends Coroutine<I,E> {
  X lazyEval(); }

public interface Coroutine<I,E> {
  Report<I,E> nextStep(); }
```

Both the node and the stepwise computations are coroutines. We can invoke a coroutine such that it runs until it yields a report. The stepwise computations orchestrate *nextStep* calls on the nodes, as we see later.

Along similar lines as the protocol that we described above, we construct the AST, assign values for the root's inherited attributes, use *begin* on the root to get its stepwise computation, then finally invoke *lazyEval* to get the synthesized attributes.

```
public final class Main {
  public static void main(final String[] args) {
    // build tree
    Pred p1 = new PredVar("x");
    Pred p2 = new PredVar("y");
    Pred p3 = new PredOr(p1, p2);

    // set inherited attributes of the root
    p3.inhs().env().set(new HashMap<String, Boolean>());
    p3.inhs().env().get().put("x", false);
    p3.inhs().env().get().put("y", true);

    // start on-demand evaluation
    Stepwise<PredSyn, Info, BacktrackException> outcome = p3.begin();
    boolean result = outcome.lazyEval().value().get();
    System.out.println("result: " + result); }}
```

Since Java does not have native support for coroutines, we encode it as an object that has a *nextStep* method that can be invoked multiple times. During each invocation it may perform some computations and yield a progress report. It may keep track of its local state via private fields of the encapsulating object.

To ease the implementation of coroutines, the *CoroutineBase* class provides boilerplate code. Such a coroutine must implement the method *visit*. This method encodes a sequence of computations indexed by a parameter *_state*. This parameter is incremented after each invocation, such that the *visit* method can invoke the next sequence of computations. The visit method does not directly return a progress report, but may call API functions to enqueue one or more progress reports. The *nextStep* method invokes the *visit* method until there is at least one element in the queue, and subsequently returns this report.

```
public abstract class CoroutineBase<I, E> implements Coroutine<I, E> {
  private LinkedList<Report<I, E>> _actions;
  private int _state;

  public CoroutineBase() {
      _state = 0;
    _actions = new LinkedList<Report<I, E>>(); }

  public Report<I, E> nextStep() {
    Report<I, E> rep = null;

    while (true) {
      rep = _actions.poll();
      if (rep == null) {
        visit(_state);
        ++_state; }
      else
        return rep; }}

  protected abstract void visit(final int state);

  protected void emit(final I info) {
    _actions.add(new ReportInfo<I, E>(info)); }

  protected void resumeAfter(final Stepwise<?, I, E> child) {
    _actions.add(new ReportChild<I, E>(child)); }

  protected void abort(final E failure) {
    _actions.add(new ReportFail<I, E>(failure)); }

  protected void done() {
    _actions.add(new ReportDone<I, E>()); }

  protected void commit(final Stepwise<?, I, E> comp) {
    _actions.add(new ReportReplace<I, E>(comp)); }}
```

The method *emit* enqueues a *ReportInfo* report that provides user-specified progress information. The method *resumeAfter* enqueues a *ReportChild* report. It demands that the *nextVisit* method is only called again after strict evaluation of the provided stepwise computation (presumably a child of the current computation) is run to completion first. The methods *abort* and *done* enqueue failure and completion reports respectively. Finally, the method *commit* enqueues a *ReportReplace* report. It specifies that the current stepwise computation should be replaced by the provided computation. We use this report later to be able to replace a choice between stepwise computations by one of the choices.

```
public interface Report<I, E> {}

public class ReportReplace<I,E> implements Report<I,E> {
  private Stepwise<?,I,E> _comp;

  public ReportReplace(final Stepwise<?,I,E> comp) {
    _comp = comp; }

  public Stepwise<?,I,E> get() {
    return _comp; }}

public class ReportInfo<I,E> implements Report<I,E> {
  private I _info;

  public ReportInfo(final I info) {
    _info = info; }

  public I get() {
    return _info; }}

public class ReportFail<I,E> implements Report<I,E> {
  private E _failure;

  public ReportFail(final E failure) {
    _failure = failure; }

  public E get() {
    return _failure; }}

public class ReportDone<I,E> implements Report<I,E> {
  public ReportDone() {}}

public class ReportChild<I,E> implements Report<I,E> {
  private Stepwise<?,I,E> _child;

  public ReportChild(final Stepwise<?,I,E> child) {
    _child = child; }

  public Stepwise<?,I,E> get() {
    return _child; }}
```

In contrast to the Haskell implementation does a *ReportDone* not provide the resulting values for synthesized attributes. This turns out to be more convenient for the Java implementation, because we then don't have to concern ourselves with the type *X* of the stepwise computations. We hide this type via existentials in the *ReportReplace* and *ReportChild* reports. After strict evaluation is done for a node, we may call *lazyEval* on the stepwise computation and get immediate access to the already evaluated attributes.

The visit methods of nodes invoke rules in a fixed order. Alternatively, when accessing attributes, rules may be invoked on-demand. To prevent double calculations and side effect, *Rules* must implement the *execute* method, which is only called from the *run* method when it has not been run yet.

```
public abstract class Rule implements Runnable {
  private boolean _hasRun;

  public Rule() {
    _hasRun = false; }
  public void run() {
    if (!_hasRun) {
      _hasRun = true;
      execute(); }}

  protected abstract void execute(); }
```

For each production, we generate a subclass of the abstract class of its nonterminal. An object of this subclass contains values of the terminal and nonterminal symbols as private fields. These values must be provided as parameters to the constructor. The constructor creates the rules for the production, and connects rules to the synthesized and local attributes. The attributes are constructed by the constructor of the *Pred* base class. A rule may refer to other attributes. If such an attribute has not been evaluated, the dereference causes its on-demand evaluation. When a rule is called via strict evaluation, it is actually guaranteed that the attributes where the rule depends on are already evaluated.

```
public final class PredVar extends Pred {
  private final String _name;  // symbols
  private final Rule _rule1;   // rules of the node

  public PredVar(final String name) {
    _name = name;

    // construct rules
    _rule1 = new Rule() {
      public void execute() {
        boolean b = inhs().env().get().get(_name);
        syns().value().set(b); }}

    // setup dependencies of synthesized and local attributes
    syns().value().dependsOn(_rule1); }

  protected void visit(final int state) {
    switch (state) {
    case 0:
      _rule1.run(); // compute syn attr
      emit(new InfoWork());
      break;
    default:
      done();
      break; }}}
```

The *visit* method executes rules in a fixed order, such that values are already computed before they are needed (strict evaluation).

Nodes are either constructed by a rule, or available as private field. A node, however, is not the same concept as a child. A child is represented by two attributes: one attribute containing a reference to the associated node, and an attribute containing the stepwise computation of a child. Virtual children are only represented by the attribute containing the stepwise computation. The former attribute is used to associate rules to the child's inherited attributes (or assign concrete values from these rules). The latter attribute is used to obtain the stepwise computation in order to access the child's synthesized attributes. Children are defined by rules (i.e. via a child-rule in the formalization). With this scheme, we support both nodes that are created up-front and nodes that can be constructed on-the-fly. For the production *PredLet* of the example, we have two nodes *_expr* and *_body*. Rules *_rule4* and *_rule5* turn these nodes into children *e* and *b* (represented by attributes *_eIn*, *_eOut*, *_bIn*, and *_bOut* respectively).

```
public final class PredLet extends Pred {
  // symbols
  private final String _name;
  private final Pred _expr;
  private final Pred _body;

  // local attributes
  private final Attr<Pred> _eIn;
  private final Attr<Pred> _bIn;
  private final Attr<Stepwise<PredSyn, Info, BacktrackException>> _eOut;
  private final Attr<Stepwise<PredSyn, Info, BacktrackException>> _bOut;

  private final Rule _rule1, _rule2, _rule3, _rule4, _rule5, _rule6, _rule7;

  public PredLet(final String name, final Pred expr, final Pred body) {
    _name = name;
    _expr = expr;
    _body = body;

    _eIn = new Attr<Pred>();
    _bIn = new Attr<Pred>();
    _eOut = new Attr<Stepwise<PredSyn, Info, BacktrackException>>();
    _bOut = new Attr<Stepwise<PredSyn, Info, BacktrackException>>();

    // construct rules
    _rule1 = new Rule() {
      public void execute() {
        HashMap<String, Boolean> env = inhs().env().get();
        _eIn.get().inhs().env().set(env); }}

    _rule2 = new Rule() {
      public void execute() {
        boolean val = _eOut.get().lazyEval().value().get();
        HashMap<String, Boolean> env = (HashMap<String, Boolean>) inhs()
            .env().get().clone();
        env.put(_name, val);
        _bIn.get().inhs().env().set(env); }}

    _rule3 = new Rule() {
      public void execute() {
        boolean val = _eOut.get().lazyEval().value().get();
        syns().value().set(val); }}

    _rule4 = new Rule() {
      public void execute() {
        _eIn.set(_expr);
        _eIn.get().inhs().env().dependsOn(_rule1); }}

    _rule5 = new Rule() {
      public void execute() {
        _bIn.set(_body);
        _bIn.get().inhs().env().dependsOn(_rule2); }}

    _rule6 = new Rule() {
      public void execute() {
        _eOut.set(_eIn.get().begin()); }}

    _rule7 = new Rule() {
      public void execute() {
        _bOut.set(_bIn.get().begin()); }}

    // setup dependencies of synthesized and local attributes
    syns().value().dependsOn(_rule3);
    _eIn.dependsOn(_rule4);
    _bIn.dependsOn(_rule5);
    _eOut.dependsOn(_rule6);
    _bOut.dependsOn(_rule7); }

  protected void visit(final int state) {
    switch (state) {
    case 0:
      _rule4.run(); // create expr child
      _rule1.run(); // assign its inh attr
      _rule6.run(); // prepare it
      resumeAfter(_eOut.get());
      break;
    case 1:
      _rule5.run(); // create body child
      _rule2.run(); // assign its inh attr
      _rule7.run(); // prepare it
      resumeAfter(_bOut.get());
      break;
    case 2:
      _rule3.run(); // assign syn attr
      done();
      break;
    default:
      done();
      break; }}}
```

In the above code, *_rule4* actually constructs the child *e* from the node stored in the private field, and associates rules to the child's inherited attributes. The *visit* method specifies the code to run until the evaluation of the first child, the code to run until the evaluation of the second child, and finally the code to run in the end. By using *resumeAfter* on a child, it ensures that the visit method is only invoked again after strict evaluation of that child is finished.

In the *PredOr* production, the choice between the two children is captured by the virtual child *res*. It is represented by an attribute that contains a stepwise computation *ChooseOr* that corresponds to this choice.

```
public final class PredOr extends Pred {
  // symbols
  private final Pred _left;
  private final Pred _right;

  // local attributes
  private final Attr<Pred> _leftIn;
  private final Attr<Pred> _rightIn;
  private final Attr<Stepwise<PredSyn, Info, BacktrackException>> _resOut;

  private final Rule _rule1, _rule2, _rule3, _rule4, _rule5, _rule6

  public PredOr(final Pred left, final Pred right) {
    _left = left;
    _right = right;

    _leftIn = new Attr<Pred>();
    _rightIn = new Attr<Pred>();
    _resOut = new Attr<Stepwise<Pred, Info, BacktrackException>>();

    // construct rules
    _rule1 = new Rule() {
      public void execute() {
        _leftIn.get().inhs().env().set(inhs().env().get()); }}

    _rule2 = new Rule() {
      public void execute() {
        _rightIn.get().inhs().env().set(inhs().env().get()); }}

    _rule3 = new Rule() {
      public void execute() {
        boolean b = _resOut.get().lazyEval().value().get();
        syns().value().set(b); }}

    _rule4 = new Rule() {
      public void execute() {
        _leftIn.set(_left);
        _leftIn.get().inhs().env().dependsOn(_rule1); }}

    _rule5 = new Rule() {
      public void execute() {
        _rightIn.set(_right);
        _rightIn.get().inhs().env().dependsOn(_rule2); }}

    _rule6 = new Rule() {
      public void execute() {
        ChooseOr choice = new ChooseOr(_leftIn.get().begin(), _rightIn
            .get().begin());
        _resOut.set(choice); }}

    // setup dependencies of synthesized and local attributes
    syns().value().dependsOn(_rule3);
    _leftIn.dependsOn(_rule4);
    _rightIn.dependsOn(_rule5);
    _resOut.dependsOn(_rule6); }}

  protected void visit(final int state) {
    switch (state) {
    case 0:
      _rule4.run(); // create left child
      _rule5.run(); // create right child
      _rule1.run(); // assign left's inh attr
      _rule2.run(); // assign right's inh attr
      _rule6.run(); // prepare it
      resumeAfter(_resOut.get());
      break;
    case 1:
      _rule3.run(); // assign syn attr
      done();
    default:
      done();
      break; }}}
```

Rule *_rule6* initializes the child by constructing a *MergeOr* computation that takes the stepwise computations of the two children as parameter. Also note that *_rule6* is assigned as dependency to *resOut* attribute. If *_rule6* is not invoked via strict evaluation, then on-demand evaluation invokes it when an attribute of child *res* is needed.

The choice between children proceeds by taking steps from both children and inspecting the progress reports. If evaluation for one of the children is finished, a choice can be made. Otherwise the choice itself emits a progress report. A choice is made via the *commit* method, which yields a *ReportReplace* report.

```
 public class ChooseOr extends Merge<Pred, Info, BacktrackException> {
   protected final Stepwise<X,I,E> _left;
```

```
  protected final Stepwise<X,I,E> _right;

  public ChooseOr(final Stepwise<Pred, Info, BacktrackException> left,
      final Stepwise<Pred, Info, BacktrackException> right) {
    _left = left;
    _right = right; }

  protected void visit() {
    Report<Info, BacktrackException> r1 = _left.nextStep();
    Report<Info, BacktrackException> r2 = _right.nextStep();

    if (r1 instanceof ReportDone)
      commit(_left.lazyEval().syns().value().get() ? _left : _right);
    else if (r2 instanceof ReportDone)
      commit(_right.lazyEval().syns().value().get() ? _right : _left);
    else if (r1 instanceof ReportFail)
      commit(_right);
    else if (r2 instanceof ReportFail)
      commit(_left);
    else if (r1 instanceof ReportInfo && r2 instanceof ReportInfo)
      emit(new InfoWork()); }}
```

The class *Merge* is a stepwise computation. It implements lazy evaluation by taking steps until the choice has been resolved. Then it proceeds with lazy evaluation on the selected child.

```
public abstract class Merge<X, I, E> extends CoroutineBase<I, E> implements
    Stepwise<X, I, E> {
  public Merge() {}

  public X lazyEval() {
    while (true) {
      Report<I, E> rep = nextStep();

      if (rep instanceof ReportReplace) {
        ReportReplace<I, E> repl = (ReportReplace<I, E>) rep;
        Stepwise<X, I, E> comp = (Stepwise<X, I, E>) repl.get();
        return comp.lazyEval(); }
      else if (rep instanceof ReportFail) {
        throw new RuntimeException(
            "all alternatives fail."); }}}}
```

Note the cooperation between stepwise and on-demand evaluation. Lazy evaluation never has to redo the work already done due to stepwise evaluations.

Finally, we show the stepwise computation that can be obtained from a node. The *Parents* class drives the stepwise evaluation of a subtree. It represents the intermediate stages of strict evaluation on this subtree. The subtree is represented by a stack. The active node is at the top of the parents stack. The bottom of the stack is the root of the subtree. Lazy evaluation returns the synthesized attributes of the root. Depending on how much strict evaluation took place, attributes may already have been computed. The *nextStep* method delegates invocations to the deepest nodes of subtrees (e.g. the top of the stack) until a progress report can be yielded. Evaluation of such a node may cause new children to be pushed on the stack (as reaction on a *ReportChild* report), nodes to be replaced (as reaction of a *ReportReplace* report), and children to be popped off (when evaluation for a child is complete).

```
public final class Parents<X,I,E> implements Stepwise<X,I,E> {
  private X _syns;
  private LinkedList<Coroutine<I,E>> _stack;

  public Parents(final Node<X,I,E> node, final X syns) {
    _stack = new LinkedList<Coroutine<I,E>>();
    _stack.add(node);
    _syns = syns; }

  public X lazyEval() {
    _stack.clear();
    return _syns; }

  public Report<I,E> nextStep() {
    while(true) {
      Coroutine<I,E> head = _stack.poll();
      if (head == null)
        return new ReportDone<I,E>(); // stack empty, we are done

      if (head instanceof Parents) {  // merge stacks
        Parents<?,I,E> other = (Parents<?,I,E>) head;
        _stack.addAll(0, other._stack);
        continue; }

      Report<I,E> rep = head.nextStep();
      if (rep instanceof ReportReplace) {
        ReportReplace<I, E> repl = (ReportReplace<I, E>) rep;
        Coroutine<I, E> comp = (Coroutine<I, E>) repl.get();
        _stack.addFirst(comp);
        continue; }
      else if (rep instanceof ReportFail)
        return rep;
      else if (rep instanceof ReportChild) {
        ReportChild<I, E> child = (ReportChild<I, E>) rep;
        Coroutine<I, E> comp = (Coroutine<I, E>) child.get();
        _stack.addFirst(head);
        _stack.addFirst(comp);
        continue; }
      else if (rep instanceof ReportDone)
        continue;
      else if (rep instanceof ReportInfo) {
        _stack.addFirst(head);
        return rep; }}}}
```

The implementation can be improved a bit further, because we do not actively remove references to children that are not needed anymore, thus retain memory longer than strictly necessary. Furthermore, *lazyEval* can be improved by returning either the values of the synthesized attributes, or a replacement computation. When a rule needs a synthesized attribute, it should (via some helper object) iterate *lazyEval* until it eliminated all intermediate *Merge* nodes. The memory usage of the bookkeeping is linear in the number of nodes of the tree plus the maximum number of progress reports that can be yielded by a node. The computations needed for the bookkeeping run in time linear in the number of nodes plus for each merge node in time linear to the maximum number of progress reports.

We showed a translation to Java. In a similar way a translation to C# can be made. An advantage of C# over Java is that local attributes without a type signature can be supported using untyped fields. In principle, the approach of this chapter is not restricted to a particular programming language. However, care has to be taken in combination with side effect. When a node is shared (i.e. as in remote reference AGs [Magnusson and Hedin, 2007]), a progress report is only received by one the parents. This behavior may be desirable; many search algorithms work on graphs instead of trees.

8 Residuation

Given a type system that is written as a collection of type rules, we investigate the automatic derivation of inference algorithms from these rules. A minor challenge are the side effects of a rule, which need to be expressed algorithmically. A major challenge are non-deterministic aspects of rules that cannot be directly mapped to an algorithm.

We present Ruler, a language for type inferencers, to meet these challenges. An inferencer is written as a collection of rules with side conditions explicitly expressed in Haskell, and with annotations for the scheduling of the rules.

This chapter includes an extensive case study of an inferencer for the "First Class Polymorphism for Haskell" type system [Vytiniotis et al., 2008].

8.1 Introduction

A type system is "a tractable syntactic method for proving the absence of certain program behaviors by classifying phrases according to the kinds of value they compute" [Pierce, 2002]. Given a type system, it is often not immediately clear whether there exists an algorithm that can automatically infer a valid type for a (type correct) program. More specifically, if the type system has principal types, is there an algorithm that can infer the most general type of each expression?

Most type systems have a declarative specification in the form of a collection of type rules. How to effectively and efficiently use these rules for building type correctness proofs is a separate issue, and having a systematic way in building such type inferencers from such a collection of rules is still an open issue. The benefits of having such a method are:

Consistency. A strong coupling between formal description and implementation makes it easier to show how certain properties proved for the type system carry over to the inferencer.

Rapid prototyping. Experimenting with an implementation of a type systems leads to a deeper understanding of the meaning of the rules and their complexity. However, language developers are currently discouraged to do so as it is cumbersome to write inferencers from scratch. A framework will relieve programmers from this burden and hence support rapid prototyping.

Abstraction. Interacting language features obscure and complicate language semantics. In many inference algorithms, the unification procedure makes the essential decisions about non-deterministic aspects. This requires context-information to be carried to and into the place where unification is performed, and complicates the inferencer. Instead, we would like to be able to deal with the decision making process at the places where the non-deterministic aspects occur in the type rules.

Documentation. In comparison with the type rules, the type inference algorithms are often not completely documented and explained. Often they are specified by a concrete (and sometimes obscure) implementation.

This chapter shows that it is possible to semi-automatically obtain inference algorithms from type rules. We do not get them entirely for free. A major problem is that type rules generally contain non-deterministic aspects. For example, more than a single rule may be applicable at the same time. Even when the rules are syntax directed, they may state demands in the form of side conditions about a type (or other value), of which concrete information is not easily available from the context.

A solution to this challenge are *annotations* which control the scheduling of the resolution of non-deterministic aspects by manipulating guesses. A *guess* is an opaque value representing a derivation that has not been constructed yet. It also serves as a place holder for a concrete value. We can pass such guesses around, observe them, and impose requirements on them. When a sufficient number of requirements have been accumulated, the actual value of the guess is revealed and we can attempt to construct the derivation.

Therefore, we contribute the following:

- We present a typed domain specific language for type inferencers called Ruler. One of its distinguishing features is the possibility to provide annotation for type rules. Also, side expressions are expressed using conventional Haskell code.
- We give examples of increasing complexity of inferencers for type systems with non-deterministic aspects, and show how manipulating guesses leads to their resolution (Section 8.2). We demonstrate the power of these annotations by providing an inferencer for the type system of FPH [Vytiniotis et al., 2008] that is directly based on FPH's collection of declarative type rules (Section 8.2.4).
- We formalize the notation (Section 8.4), the operational semantics (Section 8.5) and the static semantics (Section 8.7) of Ruler.
- We discuss the rationale of our design compared to prior work on the construction of type inferencers (Section 8.3).
- We have a proof-of-concept, Haskell-based implementation for a meta-typed front-end in which inferencer rules with custom syntax can be encoded. Furthermore we provide an executable version of the operational semantics which interprets the inference rules and produces a derivation in terms of the original type rules for all expressions it manages to type.

The reason that we have chosen Haskell as the target language for our generated inferencers are:

- We can use the expressiveness of Haskell for writing the semantics of side conditions in type rules.

- There are many libraries available for Haskell that provide efficient data structures and external constraint solvers.
- We can integrate the inferencer with other Haskell projects, in particular the Utrecht Haskell Compiler [Dijkstra et al., 2007a]. We have compiler technology readily available (parsers, tree-walk generators, pretty printers, etc.) to facilitate rapid prototyping.

8.2 Examples

In this section we show how to use Ruler in describing a series of type inferencers of increasing complexity. We took the examples such that each example builds on the previous one. We start from an inferencer for the explicitly typed lambda calculus in Section 8.2.1. Admittedly, the inferencer in this case does only type checking, but we use it to informally introduce the Ruler inferencer language (formally in Section 8.4) and informally describe its evaluation model (formally in Section 8.5). Then, in Section 8.2.2, we move on to an inferencer for implicitly typed System F, in which several cases of non-determinism arise. Finally, we show the inferencer for FPH in Section 8.2.4, which demonstrates the expressive power of the annotations.

For each example we show the type rules and the actual inferencer code. As they have a tight resemblance, be warned not to confuse the two!

Syntax:

$$\begin{array}{rcl} e & = & x \mid f\,a \mid \lambda(x :: \tau).\,e \mid \mathbf{let}\,x = e\,\mathbf{in}\,b \mid \mathbf{fix}\,f \\ \tau & = & \alpha \mid \tau_1 \to \tau_2 \end{array}$$

Rules:

$$\frac{(x,\tau) \in \Gamma}{\Gamma \vdash x : \tau}\ \textsc{var} \qquad \frac{\Gamma \vdash f : \tau_f \quad \Gamma \vdash a : \tau_a \quad \tau_f \equiv \tau_a \to \tau_r}{\Gamma \vdash f\,a : \tau_r}\ \textsc{app} \qquad \frac{x :: \tau_x, \Gamma \vdash e : \tau}{\Gamma \vdash \lambda(x :: \tau_x).e : \tau_x \to \tau}\ \textsc{lam.expl}$$

$$\frac{\Gamma \vdash e : \tau_x \quad x :: \tau_x, \Gamma \vdash b : \tau}{\Gamma \vdash \mathbf{let}\,x = e\,\mathbf{in}\,b : \tau}\ \textsc{let} \qquad \frac{\Gamma \vdash f : \tau_f \quad \tau_f \equiv \tau \to \tau}{\Gamma \vdash \mathbf{fix}\,f : \tau}\ \textsc{fix}$$

Figure 8.1: Type system for explicitly typed lambda calculus.

8.2.1 Explicitly Typed Lambda Calculus

Figure 8.1 gives the type system for the explicitly typed lambda calculus [Church, 1940, Pierce, 2002]. The rules define a *relation* between an environment Γ, expression e, and type τ.

For the inferencer, this corresponds to a *function* (we call it a *scheme*) that takes an environment Γ and expression e as inputs and produces a valid type τ, if there exist such a type according to the rules. The Ruler code of the inferencer of this type system is given in Figure 8.2. We discuss each part further below.

Scheme declarations. To obtain an inferencer in Haskell, we actually want a Haskell function *tc* with the type: *Map String Ty* $\rightarrow$ *Expr* $\rightarrow$ *I Ty* where *I* is some monad encapsulating failure and state. Thus, concerning the meta variables, we need to know whether they serve as inputs or outputs, and what their meta type is. This information is given in the scheme declaration. It defines the name of the function (i.e. *tc*), the syntax of the function call in the inferencer rules (scheme instantiation), the names and types of the meta variables, whether a meta variable is an input ($\lhd$) or an output ($\rhd$), and a optional property d or u of a meta variable. In this case, the d-property requires that we supply an instance of *Deferrable* for the Haskell type *Ty*, and allows us to use the defer statement on types (to be explained later).

Inferencer rules. The inferencer rules provide the actual definition of the scheme. They consist of an ordered sequence of *statements*, related to the *premises* of the type rules, and a concluding statement. Such a statement can be:

- A scheme invocation, i.e. $(x,\tau) \in \Gamma$, which executes the corresponding function with the given parameters when evaluated.
- Haskell code in the *I* monad. This code is used to express side-conditions of type rules as statements in the inferencer rules. For example, the type system in Figure 8.2 implicitly mentions a lookup-relation in the VAR rule. This is explicitly defined in our inferencer code by means of some Haskell code in the LOOKUP rule.
- An equality statement, i.e. $\tau_f \equiv \tau_a \rightarrow \tau_r$, stating that its two inputs will be the same after type inference has finished.
- Non-determinism annotations, such as defer (explained later).

The rules represent the actual definition of cases for functions *tc*, *lk*, and *fr*. For example, the APP, LOOKUP, and FRESH inferencer rules are projected to concrete Haskell code as follows:

```
tc_app Γ e                          lk_lookup x Γ
    = do let (EApp f a) = e           = do v ← lookup x Γ
            τ_f ← tc Γ f                     return v
            τ_a ← tc Γ a               lk = lk_lookup
            τ_r ← fr                   fr_fresh = do (v, ()) ← defer f
            unif τ_f (TArr τ_a τ_r)                 return v
            return τ_r                 where f v' = return ()
tc = tc_var ⊕ tc_app ⊕ ...           fr = fr_fresh
```

Scheme declarations:

$$\begin{array}{lll}
\mathsf{tc}: & & (\Gamma \lhd \mathsf{Map\,String\,Ty}) \vdash (e \lhd \mathsf{Expr}) \;:\; (\tau \rhd_{\mathsf{d}} \mathsf{Ty}) \\
\mathsf{fr}: & \forall\alpha. & (v \rhd_{\mathsf{d}} \alpha)\ \text{fresh} \\
\mathsf{lk}: & \forall\alpha\beta. & (\,(k \lhd \alpha)\,,\,(v \rhd \beta)\,) \in (\Gamma \lhd \mathsf{Map}\,\alpha\,\beta)
\end{array}$$

Inferencer rules:

$$\frac{(x,\tau) \in \Gamma}{\Gamma \vdash x : \tau}\,\textsc{var} \qquad \frac{v \leftarrow \mathit{lookup}\ x\ \Gamma}{(x,v) \in \Gamma}\,\textsc{lookup} \qquad \frac{\Gamma \vdash f : \tau_f \quad \Gamma \vdash a : \tau_a \quad \tau_r\ \text{fresh} \quad \tau_f \equiv \tau_a \to \tau_r}{\Gamma \vdash f\,a : \tau_r}\,\textsc{app}$$

$$\frac{\mathsf{defer}_v\ [\emptyset]}{v\ \text{fresh}}\,\textsc{fresh} \qquad \frac{x :: \tau_x, \Gamma \vdash e : \tau}{\Gamma \vdash \lambda(x :: \tau_x).e : \tau_x \to \tau}\,\textsc{lam.expl}$$

$$\frac{\Gamma \vdash e : \tau_x \quad x :: \tau_x, \Gamma \vdash b : \tau}{\Gamma \vdash \mathbf{let}\ x = e\ \mathbf{in}\ b : \tau}\,\textsc{let} \qquad \frac{\tau\ \text{fresh} \quad \Gamma \vdash f : \tau_f \quad \tau_f \equiv \tau \to \tau}{\Gamma \vdash \mathbf{fix}\ f : \tau}\,\textsc{fix}$$

Syntax and semantics:

```
data Ty = TGuess GuessVar   data Expr = EVar String
        | TConst GuessVar             | EApp Expr Expr
        | TArr    Ty  Ty              | ...
instance Container Ty where
  appSubst rec (TArr f a) = rec f ≫ rec a
  deferVars (TConst _)    = empty
  deferVars (TGuess v)    = single v
  deferVars (TArr a r)    = deferVars a `union` deferVars r
instance Unifyable Ty where
  unify rec (TArr f a) (TArr g b) = rec f g ≫ rec a b
  unify _   _          _          = fail "type error"
instance Deferrable Ty where
  mkDeferValue                  = TGuess
  mkFixedValue                  = TConst
  matchDeferValue (TGuess v) = Just v
  matchFixedValue (TConst v)  = Just v
pattern Map String Ty where  x :: τ, Γ input   insert x τ Γ
pattern Expr          where  λx . e   output ELam x e
  -- other patterns omitted.
```

Figure 8.2: Inferencer for explicitly typed lambda calculus.

The equivalence statement gets translated to a monadic expression **unif** which is an API function provided by Ruler.

The statements are executed in the order of appearance. Each statement may fail, causing the entire rule to fail. Rules that fail due to pattern matches or equality statements at the very beginning of the statement sequence allow other applicable rules to be applied (offering a limited form of backtracking). Otherwise, the failure is turned into an abort of the entire inference, with a type error as result.

Non-determinism. Not all relations that occur in a type rule are functions. Sometimes a meta variable is required to be both an input and an output. For example, in the inferencer rule APP, the value τ_r needs to be produced before it can be passed to the equality statement. It is also an output of the rule, not an input. This means that there is no indication how to obtain it. Therefore, we conceptually *guess* the value of τ_r. This value is kept hidden behind an opaque guess-value, and is only revealed when we actually discover what the value must be. The *fr*-scheme gives a function that produces these values.

Ruler accomplishes this as follows. The rule FRESH has a defer-statement, which is a non-determinism annotation. It is parametrized with a list of statement sequences, and produces a guess v. The statement sequences are not executed immediately, but a closure is created for them which is triggered once we discover concrete information about guess v. At that point, one of the statement sequences is required to execute successfully with v as an input and the current knowledge about guesses. The defer-statement in FRESH has only one statement sequence, the empty sequence, which always succeeds. In later examples we have non-trivial sequences of statements that allow us to defer and control decision making.

So, a guess needs to get produced for v. Ruler requires help in the form of a *Deferrable* instance on the type of v to construct this guess. Operationally, defer produces an opaque guess variable, which is wrapped into the domain of v by means of *mkDeferValue*. This guess is thus first class, and can be passed around and end up in other data structures. Ruler maintains information about these guess variables, such as the closures produced by defer. When a concrete value is discovered about a guess, all occurrences of this guess are replaced with this concrete value (thus revealing the guess). Again, Ruler requires help by means of a *Container* instance in order to deal with values holding guesses. Furthermore, the inferencer rules may check if certain values are still opaque variables and act on that. This is also something we exploit later.

Concrete values for a guess are discovered by executing equality statements. When comparing the two input values, if one value contains a guess and the other a concrete value, then we *commit* that concrete value to the guess. This leads to the execution of the deferrable statements. We call this commit because it is an irreversible action: a guess can be opaque for a while, a commit conceptually only uncovers it. If both values are guesses, the guesses are merged. In case both values are concrete values, Ruler requires help in the form of a *Unifyable* instance, of which *unify* is required to traverse one level through the values and check that their heads are the same. Another requirement on guesses is that the value committed to a guess may not contain the guess itself (the infamous *occur check*). Therefore, the function *deferVars* needs to be defined to tell Ruler which guesses are contained in a value.

Data semantics. The last part of the Ruler code consists of a definition of the data structures involved. One may also define custom syntax to be used in the rules, for which translations to either Haskell patterns (for inputs) or Haskell expressions (for outputs) need to be given. This custom syntax may be ambiguous as long as it can be resolved based on the meta types of the meta variables. Finally, we remark that these instances for data types are likely to be automatically generated from the structure of the data types, or readily available in a library with support code.

Before we continue, consider the addition of an extra rule to the inferencer:

$$\frac{\tau_x \text{ fresh} \qquad x :: \tau_x, \Gamma \vdash e : \tau}{\Gamma \vdash \lambda x.e : \tau_x \rightarrow \tau}\ \text{LAM.IMPL}$$

With this addition, we obtain an inferencer for the simply typed lambda calculus. However, with this rule, there can be unresolved guesses remaining after we finish inferencing. For example, consider inferring the type of the expression $\lambda x\ .\ x$. We obtain $\tau \rightarrow \tau$, where τ is a guess. However, at the end of the inference, Ruler forces all remaining guesses to get evaluated. Those guesses that remain are mapped to a *fixed* value. A fixed value is an opaque value like a guess (created with *mkFixedValue*), except that it is only equal to itself and cannot be committed on.

8.2.2 Implicitly Typed System F

We give a sound but incomplete inferencer for implicitly typed System F [Reynolds, 1974], using a relaxation of Milner's algorithm [Milner, 1978] and exploiting type annotations to deal with higher-ranked types. Compilers such as GHC and UHC utilize inference algorithms based on this type system, which makes it an interesting case study. The inferencer algorithm described here is clearly inferior versus other algorithms in terms of completeness and predictability, but is powerful and simple enough to serve as a basis for the inference algorithm of the next section.

Extra syntax:

$$\tau = \ldots \mid \forall \overline{\alpha}.\, \tau$$

Extra rules:

$$\frac{\Gamma \vdash e : \forall \overline{\alpha}.\, \tau_2}{\Gamma \vdash e : [\overline{\alpha := \tau_1}]\, \tau_2}\ \text{INST} \qquad\qquad \frac{\Gamma \vdash e : \tau_1 \qquad \overline{\alpha} \notin \mathit{ftv}\, \Gamma}{\Gamma \vdash e : \forall \overline{\alpha}.\, \tau_1}\ \text{GEN}$$

Figure 8.3: Type system of implicitly typed System F.

Implicitly typed System F (or polymorphic lambda calculus) extends the simply typed lambda calculus with two rules, and a more expressive type language (Figure 8.4). This change adds a lot of expressive power. Consider the following expressions (assuming for the moment that we have *Int*s in the type language):

$$\begin{aligned} f &= \lambda(k :: (Int \to Int) \to Int)\ . && k\ (\lambda x\ .\ x) \\ g &= \lambda(k :: \forall\alpha\ .\ (\alpha \to \alpha) \to Int)\ . && k\ (\lambda x\ .\ x) \end{aligned}$$

The definition of f can be typed within the simply typed lambda calculus, but g cannot. In g's case, the GEN rule is needed after typing $\lambda x\ .\ x$.

There is no simple way to translate these rules to the type inferencer rules. The problem lies in the decision when to apply these rules, because this is not specified by the syntax. They could be applied any time, even an arbitrary number of times. However, we choose to only apply instantiation once directly after the VAR rule, and generalization once for each let-binding, and once for the argument of each application. Figure 8.4 lists the inferencer rules. We partitioned the rules such that they belong either to scheme $\vdash$, $\vdash_x$, $\vdash_g$, or $\vdash_l$, and adapted the recursive invocations accordingly. This solves the problem of when to apply the rules. Also note that we have two versions of the GEN rule, one for the let-binding (GEN.LET), and one for the argument of an application (GEN.LAZY).

Inferencer rules:

$$\frac{(x,\tau) \in \Gamma}{\Gamma \vdash_x x : \tau}\ \text{VAR} \qquad \frac{\Gamma \vdash_x x : \forall\overline{\alpha}.\,\tau_2 \qquad \overline{\tau_1}\ \text{fresh}}{\Gamma \vdash x : [\overline{\alpha := \tau_1}]\,\tau_2}\ \text{INST.V}$$

$$\frac{\Gamma \vdash e : \tau_1 \qquad \mathsf{defer}_{\tau_2}\ [[\ \mathsf{let}\,(\forall\overline{\alpha}.\tau_2') = \tau_2,\ \tau_1 \equiv \tau_2',\ \overline{\alpha} \notin ftv\,\Gamma\ ]]}{\Gamma \vdash_g e : \tau_2}\ \text{GEN.LAZY}$$

$$\frac{\begin{array}{c}\tau_r\ \text{fresh} \\ \Gamma \vdash f : \tau_f \qquad \Gamma \vdash_g a : \tau_a \\ \tau_f \equiv \tau_a \to \tau_r\end{array}}{\Gamma \vdash f\,a : \tau_r}\ \text{APP} \qquad \frac{\begin{array}{c}\mathsf{fixate}_\tau\,[\,\Gamma \vdash e : \tau\,] \\ \mathsf{let}\,\overline{\alpha} = ftv\,\tau - ftv\,\Gamma\end{array}}{\Gamma \vdash_l e : \forall\overline{\alpha}.\,\tau}\ \text{GEN.LET} \qquad \frac{\begin{array}{c}\Gamma \vdash_l e : \tau_x \\ x :: \tau_x, \Gamma \vdash b : \tau\end{array}}{\Gamma \vdash \mathbf{let}\,x = e\,\mathbf{in}\,b : \tau}\ \text{LET}$$

Syntax and semantics:

$$\mathbf{data}\ Ty = \ldots \quad | \quad TAll\ [GuessVar]\ Ty$$

$$\begin{array}{ll} ftv\ (TConst\ v) = single\ v & ftv\ (TArr\ a\ r) = ftv\ a\ \text{`}union\text{`}\ ftv\ r \\ ftv\ (TGuess\ \_) = empty & ftv\ (TAll\ vs\ t) = ftv\ t\ \text{`}difference\text{`}\ vs \end{array}$$

Figure 8.4: Inferencer for implicitly typed System F.

However, this leads us back to the non-determinism problems that we encountered before. The INST.V rule requires us to choose which bound variables to instantiate, and what type to instantiate them to. Similarly, for both GEN rules, a decision needs to be made what variables to generalize over. These are all examples of non-deterministic aspects. We use the following tricks to resolve them:

- Instantiation (rule INST.V) is greedy and instantiates all bound variables that are know at the time when instantiation is applied. We use the *fr*-relation to guess the types to which they are instantiated.

- Generalization of the argument of an application (rule GEN.LAZY) is done on-demand. The result type τ_2 is guessed. At some point the head (or more) of τ_2 is discovered. In case of our example: for *f* we discover at some point that τ_2 is $Int \to Int$, and for *g* that it is $\forall\alpha.\alpha \to \alpha$. At that moment the deferred statements are triggered.

  When these statements trigger, the requirement is that enough information about the outermost quantifiers of τ_2 is known. Furthermore, with the greedy assumption about instantiation, assume that τ_1 does not have any outermost quantifiers. With this knowledge in mind, consider the GEN type rule again in Figure 8.3. The type rule tells us to take the portion of τ_1 without outermost quantifiers, which should then be equal to τ_2. This relation is kept until the end of the inference. In that case, the variables $\overline{\alpha}$ are not allowed to be in the environment. This is exactly what the deferred statements of GEN.LAZY establish.

- Generalization just before the let-binding is also greedy. It generalizes over all unbound variables in the type that are not in the environment. However, since a guess can represent an arbitrary type, we cannot generalize over them. Therefore, we introduce the fixate-statement. It is parametrized with a sequence of statements, and executes those. The guesses which are introduced during the execution and remain are forced to be evaluated. Those for which no concrete value is discovered are mapped to fixed types (*TConst* values). The order of this forcing is undefined. These *TConst* values are real type variables and can be generalized over (if free in the environment).

$$\frac{\begin{array}{c} q, \tau_x \text{ fresh} \\ \mathsf{fixate}\left[\begin{array}{l} \mathsf{defer}\_\,[\tau_x \equiv \mathit{pick}\,q] \\ (x,\tau_x,q),\Gamma \vdash e : \tau \end{array}\right] \end{array}}{\Gamma \vdash \lambda x.e : \tau_x \to \tau} \qquad \frac{\begin{array}{c} (x,\tau_1,q) \in \Gamma \\ \mathsf{defer}_{\tau_2}[q' \text{ fresh}, \mathsf{commit}_{(\mathit{last}\,q)}\,(\tau_2,q')] \\ \mathsf{defer}_{\tau_1'}\,[\,\tau_1' \leqslant \tau_2\,] \qquad \tau_1 \equiv \tau_1' \end{array}}{\Gamma \vdash x : \tau_2}$$

We defer the instantiation $\leqslant$ until we have more information about all the types we want to instantiate the left-hand side to. Queue q is a nested product, where each left component is a type, and each right component is a queue. This queue is terminated with a guess. The queue stores all encountered values for the type of the lambda parameter. Each time such a value is encountered, it gets appended to the queue. When fixating the lambda term, the deferred statement executes that traverses the queue and picks out the most general type, and matches it with the type of the lambda parameter. This causes all deferred instantiations $\leqslant$ to execute. The $\leqslant$ relation is not affected by this complex scheduling.

Figure 8.5: Complex example: queuing all expected types.

Many variants of the above rules are possible that result in a more complete inference algorithm (although no complete inference algorithm exists). For example, making instantiation also happen on demand, or queuing up all guesses of the type of lambda parameters (see Figure 8.5) before making a final choice, thus emulating type propagation algorithms [Peyton

Types with boxes:

$$\tau = \ldots \mid \boxed{\tau}$$

Type rules:

$$\frac{\Gamma \vdash x : \forall\overline{\alpha}.\,\tau_2 \qquad \tau_1 \text{ unboxed iff mono}}{\Gamma \vdash x : [\boxed{\overline{\alpha} := \tau_1}]\,\tau_2}\ \textsc{inst} \qquad \frac{\Gamma \vdash e : \tau_1 \qquad \tau_1 \preceq\sqsubseteq \tau_2}{\Gamma \vdash e : \tau_2}\ \textsc{subs} \qquad \frac{\Gamma \vdash f : \tau_f \quad \Gamma \vdash a : \tau_a \quad \lfloor\tau_a\rfloor \equiv \lfloor\tau_a'\rfloor \quad \tau_f \equiv \tau_a' \to \tau_r}{\Gamma \vdash f\,a : \tau_r}\ \textsc{app}$$

$$\frac{\Gamma \vdash e : \tau_1 \qquad \lfloor\tau_1\rfloor \equiv \tau_2}{\Gamma \vdash (e :: \tau_2) : \tau_2}\ \textsc{ann} \qquad \frac{\mathit{mono}\,\tau_x \qquad x :: \tau_x, \Gamma \vdash e : \tau}{\Gamma \vdash \lambda x.e : \tau_x \to \tau}\ \textsc{lam.impl} \qquad \frac{x :: \tau_x, \Gamma \vdash e : \tau}{\Gamma \vdash \lambda(x :: \tau_x).e : \tau_x \to \tau}\ \textsc{lam.expl}$$

$$\frac{\Gamma \vdash e : \tau_x \qquad \mathit{noBoxes}\,\tau_x \qquad x :: \tau_x, \Gamma \vdash b : \tau}{\Gamma \vdash \textbf{let}\,x = e\,\textbf{in}\,b : \tau}\ \textsc{let}$$

Boxy instantiation rules:

$$\boxed{\forall\overline{\alpha}.\tau_1} \preceq \boxed{[\boxed{\overline{\alpha} := \tau_2}]\tau_1}\ \textsc{bi} \qquad \tau \preceq \tau\ \textsc{br}$$

Protected unboxing rules:

$$\frac{\mathit{mono}\,\tau}{\boxed{\tau} \sqsubseteq \tau}\ \textsc{tbox} \qquad \tau \sqsubseteq \tau\ \textsc{refl} \qquad \frac{\tau_1 \sqsubseteq \tau_2 \qquad \mathit{unboxed}\,\overline{\alpha}\,\tau_1 \qquad \mathit{unboxed}\,\overline{\alpha}\,\tau_2}{\forall\overline{\alpha}.\tau_1 \sqsubseteq \forall\overline{\alpha}.\tau_2}\ \textsc{poly} \qquad \frac{\boxed{\tau_1} \sqsubseteq \tau_3 \qquad \boxed{\tau_2} \sqsubseteq \tau_4}{\boxed{\tau_1} \to \boxed{\tau_2} \sqsubseteq \tau_3 \to \tau_4}\ \textsc{conbox}$$

$$\frac{\tau_1 \sqsubseteq \tau_3 \qquad \tau_2 \sqsubseteq \tau_4}{\tau_1 \to \tau_2 \sqsubseteq \tau_3 \to \tau_4}\ \textsc{cong}$$

Figure 8.6: The FPH type system.

Jones et al., 2007, Dijkstra and Swierstra, 2006a]. Such algorithms are normally very hard to implement, because with conventional approaches the unification algorithm has to deal with it all. However, with Ruler, such algorithms can now be easily described , and locally at the places in the rules where full context-information is available, with only minimal effects on modularity.

8.2.3 Summary

We have seen several examples of non-determinism that are problematic when writing an inference algorithm. In the end, these problems boil down to deferring decisions and controlling the decision process. We have seen and will see the following annotations to resolve them:

defer introduces a guess with the promise that a sequence of statements will be executed when the guess is revealed.

fixate introduces a scope for guesses and forces all guesses introduced in this scope to be resolved when leaving the scope.

commit unveils the guess, causing the deferred statements to run.

force is syntactic sugar for a commit with a fresh guess, followed by a commit with a fixed value if the result was still a guess.

The driving force behind propagating the type information that slowly comes available are the equality statements ($\equiv$).

8.2.4 Example: FPH

The FPH type system [Vytiniotis et al., 2008] is a restriction of implicitly typed System F, such that there exist a principle type for each binding, and a complete inference algorithm that finds these types. In this section, we give an alternative inference algorithm. Comparing FPH's declarative rules (Figure 8.6) with the inference algorithm (Figure 8.7 and Figure 8.7) shows how close the resemblance is.

Consider the following example where *choose* is instantiated predicatively and impredicatively:

$f = choose\ id$
$f :: \forall \alpha\ .\ (\alpha \to \alpha) \to (\alpha \to \alpha)$ -- predicative inst
$f :: (\forall \alpha\ .\ \alpha \to \alpha) \to (\forall \alpha\ .\ \alpha \to \alpha)$ -- impredicative inst

The observation underlying FPH is that impredicative instantiation may result in more than one incomparable most general System F type for a binding. This is undesired for reasons of modularity and predictability. FPH dictates that impredicative instantiation is forbidden if it has influence on the type of a binding.

To formalize this difference, FPH introduces the concept of a box. When a bound variable is instantiated with a polymorphic type in FPH, it is enclosed within a box. A box expresses that the type it encloses may have been obtained through impredicative instantiation. FPH forbids the type of a binding to have a box in the type, thus ensuring that these possible undesired effects have no influence on the type. This absence of boxes can arise due to two reasons:

- The type with the box is simply not part of the type of the binding.
- There is an unboxing relation ($\preceq\sqsubseteq$) that allows shrinking of boxes over the monomorphic spine of a type. When we discover that the type in the box cannot influence the type of the result, we can remove the box.
- The programmer can give an explicit type signature, which does not have boxes.

A particular invariant maintained by FPH is that there may not be a box within a box (the "monomorphic substitution" operator $:=$ takes care of this).

The type rules for FPH contain many non-deterministic aspects, especially due to the interaction between types and boxes. Both the structure of the types, and the demands on the boxes become only gradually available. In some cases, we may discover that the type is not allowed to have any boxes before the actual type becomes known. Alternatively, in case of box-stripping ($\lfloor\cdot\rfloor$), we may know portions of the type structure, but nothing about the boxes yet.

Boxy types:

```
type Ty   = (Ty', Box)
data Box  = BYes | BNo | BVar GuessVar
data Ty'  = TArr Ty' Ty' | TGuess GuessVar | ...
```

Scheme declarations:

$$(\tau_1 \lhd_{\mathsf{d}} Ty) \sqsubseteq (\tau_2 \rhd_{\mathsf{d}} Ty) \qquad (\tau_1 \lhd_{\mathsf{d}} Ty) \sqsubseteq' (\tau_2 \lhd_{\mathsf{d}} Ty)$$
$$(\tau_1 \lhd_{\mathsf{d}} Ty) \preceq (\tau_2 \rhd_{\mathsf{d}} Ty) \qquad (\tau_1 \lhd_{\mathsf{d}} Ty) \preceq' (\tau_2 \rhd_{\mathsf{d}} Ty)$$

Inferencer rules:

$$\frac{\begin{array}{l}\Gamma \vdash_x x : \forall \overline{\alpha}.\, \tau_2 \qquad \overline{\tilde{\tau}_1}\ \text{fresh} \\ \qquad [\, [\![v]\!] = b,\ \mathsf{commit}_v\, BYes \,], \\ \mathsf{defer}_{b_i}\ [\, b = BNo,\ mono\ \tilde{\tau} \,], \\ \qquad [\, b = BYes \,]\end{array}}{\Gamma \vdash x : [\overline{\alpha := \boxed{\tilde{\tau}_1}_b}]\, \tau_2}\ \text{INST.V} \qquad \frac{\begin{array}{c}\Gamma \vdash e : \tau_1 \\ \tau_1 \preceq\sqsubseteq \tau_2\end{array}}{\Gamma \vdash_s e : \tau_2}\ \text{SUBS} \qquad \frac{\begin{array}{c}\Gamma \vdash_s e : \tau_1 \\ \ldots\end{array}}{\Gamma \vdash_g e : \tau_2}\ \text{GEN.LAZY}$$

$$\frac{\begin{array}{cc}\Gamma \vdash_s f : \tau_f & \Gamma \vdash_g a : \tau_a \\ \lfloor \tau_a \rfloor \equiv \lfloor \tau'_a \rfloor & \tau_f \equiv \tau'_a \to \tau_r\end{array}}{\Gamma \vdash f\, a : \tau_r}\ \text{APP} \qquad \frac{\begin{array}{c}\Gamma \vdash_g e : \tau_1 \\ \lfloor \tau_1 \rfloor \equiv \tau_2\end{array}}{\Gamma \vdash (e :: \tau_2) : \tau_2}\ \text{ANN}$$

$$\frac{\begin{array}{c}\mathsf{fixate}_b\, [\, \Gamma \vdash_l e : \tau_x,\ noBoxes\ \tau_x \,] \\ x :: \tau_x, \Gamma \vdash b : \tau\end{array}}{\Gamma \vdash \mathbf{let}\ x = e\ \mathbf{in}\ b : \tau}\ \text{LET}$$

Boxy instantiation rules:

$$\frac{b_2\ \text{fresh} \qquad \mathsf{defer}_{\tilde{\tau}_2}\, [\, b_1 \equiv b_2,\ \mathsf{force}\, b_1,\ \boxed{\tilde{\tau}_1}_{b_1} \preceq' \tau,\ \tau \equiv \boxed{\tilde{\tau}_2}_{b_2} \,]}{\boxed{\tilde{\tau}_1}_{b_1} \preceq \boxed{\tilde{\tau}_2}_{b_2}}\ \text{BOXY.INST}$$

$$\frac{\overline{\tau_2}\ \text{fresh}}{\boxed{\forall \overline{\alpha}.\tau_1} \preceq' \boxed{[\overline{\alpha := \tau_2}]\tau_1}}\ \text{BI} \qquad \tilde{\tau} \preceq' \tilde{\tau}\ \text{BR}$$

Figure 8.7: Inferencer for FPH (part 1).

Idea. We choose here to be able to guess the type independent of the boxes. Each alternative of a type gets a box annotation. Types τ are of the form $\boxed{\tilde{\tau}}_b$, where $\tilde{\tau}$ is a regular type with types τ as components, and b a box annotation. A box annotation is either concrete (*BYes* or *BNo*), or a guess. Types in the environment have box-annotations *BNo*, and box-annotations *BYes* are introduced by instantiation. The unboxing rules then relate boxes to types, and eliminate boxes as soon as more type information becomes available. This unboxing (see subscript s and SUBS) is done for each sub-expression. For an application, only the instantiation of the function type can cause boxes to appear in the result type. Possible boxes in the type of the argument are stripped away (these would otherwise cause boxes inside boxes). Also, annotations are considered safe and causes boxes to disappear. Finally, at a let-binding, we first generalize and fixate the guesses in the types, and only then fixate the boxes. This ensures that when fixating the boxes, we know that choices of the boxes do not influence the types in the local scope anymore.

Protected unboxing rules:

$$\frac{\tilde{\tau}_2\ \text{fresh} \qquad \lfloor\tilde{\tau}_1\rfloor \equiv \lfloor\tilde{\tau}_2\rfloor \qquad \mathsf{defer}_{b_2}\ \begin{array}{l}[\ \mathsf{let}\ [\![\_]\!] = b_2,\ b_1 \equiv b_2,\ \mathsf{force}\,b_2,\ \boxed{\tilde{\tau}_1}_{b_1} \sqsubseteq' \boxed{\tilde{\tau}_2}_{b_2}\ ],\\ [\ \mathsf{let}\ [\![v]\!] = b_1,\ \mathsf{commit}_v\,b_2,\ \boxed{\tilde{\tau}_1}_{b_1} \sqsubseteq' \boxed{\tilde{\tau}_2}_{b_2}\ ],\\ [\ \boxed{\tilde{\tau}_1}_{b_1} \sqsubseteq' \boxed{\tilde{\tau}_2}_{b_2}\ ]\end{array}}{\boxed{\tilde{\tau}_1}_{b_1} \sqsubseteq \boxed{\tilde{\tau}_2}_{b_2}}$$

$$\frac{\begin{array}{c}\mathsf{let}\ [\![v]\!] = \tilde{\tau}_1\\ \mathsf{defer}_{\tilde{\tau}'_1}\ [\boxed{\tilde{\tau}_1}_b \sqsubseteq' \tau_2]\\ \tilde{\tau}_1 \equiv \tilde{\tau}'_1\end{array}}{\boxed{\tilde{\tau}_1}_b \sqsubseteq' \tau_2}\ \text{UNBOX.TY.DEFER} \qquad \frac{\mathit{mono}\ \tilde{\tau}}{\boxed{\tilde{\tau}} \sqsubseteq' \tilde{\tau}}\ \text{TBOX} \qquad \boxed{\forall\overline{\alpha}.\tau}_b \sqsubseteq' \boxed{\forall\overline{\alpha}.\tau}_b\ \text{REFL}$$

$$\frac{\begin{array}{ccc}\tau_1 \not\equiv \tau_2 & \tau_1 \sqsubseteq \tau'_2 & \tau'_2 \equiv \tau_2\\ \mathit{unboxed}\,\overline{\alpha}\,\tau_1 & & \mathit{unboxed}\,\overline{\alpha}\,\tau_2\end{array}}{\forall\overline{\alpha}.\tau_1 \sqsubseteq' \forall\overline{\alpha}.\tau_2}\ \text{POLY} \qquad \frac{\begin{array}{cc}\boxed{\tilde{\tau}_1} \sqsubseteq \tau_3 & \tau_3 \equiv \tau'_3\\ \boxed{\tilde{\tau}_2} \sqsubseteq \tau_4 & \tau_4 \equiv \tau'_4\end{array}}{\boxed{\tilde{\tau}_1} \to \boxed{\tilde{\tau}_2} \sqsubseteq' \tau'_3 \to \tau'_4}\ \text{CONBOX}$$

$$\frac{\begin{array}{cc}\tilde{\tau}_1 \sqsubseteq \tau_3 & \tau_3 \equiv \tau'_3\\ \tilde{\tau}_2 \sqsubseteq \tau_4 & \tau_4 \equiv \tau'_4\end{array}}{\tilde{\tau}_1 \to \tilde{\tau}_2 \sqsubseteq' \tau'_3 \to \tau'_4}\ \text{CONG}$$

Semantics:

instance *Unifyable Box* **where** ...
instance *Deferrable Box* **where** ... *mkFixedValue* = *const BNo*

Figure 8.8: Inferencer for FPH (part 2).

Note the following notation. A $[\![v]\!]$ represents a guess containing variable v (produced or obtained by means of *mkDeferValue* or *matchDeferValue* respectively). A type $\tilde{\tau}$ (*Ty*′) at a place where a τ (*Ty*) is expected represents a pair of $\tilde{\tau}$ with a box annotation of *BNo*. A type

$\boxed{\tilde{\tau}}$ represents a pair of $\tilde{\tau}$ with a box annotation of *BYes*.

The boxy-instantiation rules allows instantiation inside an outermost box. The application of these rules is controlled by BOXY.INST, as follows. Decisions are delayed until more is known about the result type. Then we force the decisions to have been made about a potential box surrounding it. We then know which one the two actual instantiation rules is applicable. Note that we are not afraid of instantiation: the rule GEN.LAZY generalizes again if needed.

For protected unboxing, rule UNBOX controls how the rules are applied. It ensures that the input and output type are matched together, disregarding boxes such that this information flow is independent of the information flow about boxes. Then, applying the actual rules is deferred until the two box-annotations have been resolved. In case b_2 is still unknown, we default it. If b_2 is known, but b_1 is not, we apparently have freedom in the choice and choose b_2 for b_1. Finally, if we are in the situation that we know the annotations but not the type, we delay resolving the unboxing until we know the type by means of UNBOX.TY.DEFER.

The monadic Haskell expressions used in the premises of the inferencer rules are given in Figure 8.9. *noBoxes* forces the absence of boxes everywhere in the type, and *unBoxed* only on the spine to each occurrence of type variable a. The most involving, however, is box-stripping. It produces a type with all boxes removed, without affecting the original box annotations. The difficulty is that the type may not be fully known yet. Fortunately, we can use **defer**, *equal* and **commit** in monadic expressions too. In fact, we can write higher-order functions to factor out some patterns. For example, *dwrap* (Figure 8.10) factors out all the non-determinism of a recursive function where the input is equal to the output modulo some guesses.

Other type systems. The syntax of the defer-statement is actually a bit more general than we presented in these examples. In the examples, the deferred statements did not have outputs, only inputs. We allow the deferred statements to have outputs. For example, another type system for first class polymorphism, HML [Leijen, 2009], requires deferred statements that produce a prefix Q (denoted with $\mathsf{defer}^{Q}_{\tau}[\ldots]$).

Although our examples were about inferencers for type systems dealing with polymorphism, we stress that these were chosen in order to pave the way to a complex example, and that we are not limited to such type systems.

8.3 Related Work

We present an extension of our previous work on the Ruler system [Dijkstra and Swierstra, 2006b]. In this system, type rules are required to be written as deterministic functions, and both a type-setted LaTeX document and an efficient Attribute-Grammar based inferencer are derived from them. One of the goals of this system is to close the gap between formal description and implementation. However, non-deterministic aspects cannot be directly described in this system, and are omitted (to be solved externally). This left a gap that we attempted to close in this paper.

Box operations:

```
$noBoxes :: Ty \to I\ ()$
$noBoxes\ (t,b) = \textbf{do unif}\ b\ BNo$
                        $traverse\ noBoxes\ t$


$unboxed :: GuessVar \to Ty \to I\ ()$
$unboxed\ a\ (t,b)\ |\ a \in ftv\ t\ \ \ = \textbf{do unif}\ b\ BNo$
                                        $traverse\ (unboxed\ a)$
                  $|\ otherwise = \textbf{return}\ ()$


$\lfloor\tau\rfloor = strip\ \tau$
$strip :: Ty \to Ty$
$strip\ (t,\_) = \textbf{do}\ t' \leftarrow dtraverse\ strip\ t; \textbf{return}\ (t', BNo)$


$dtraverse\ f = dwrap\ (traverse\ f)$
$traverse\ f\ (TArr\ t_1\ t_2) = \textbf{do}\ t_3 \leftarrow f\ t_1; t_4 \leftarrow f\ t_2$
                               $\textbf{return}\ (TArr\ t_3\ t_4)$
$traverse\ f\ (TAll\ vs\ t_1)\ = \textbf{do}\ t_2 \leftarrow f\ t_1; \textbf{return}\ (TAll\ vs\ t_2)$
$traverse\ f\ t$              $= \textbf{return}\ t$


$mono\ t = \textbf{unif}\ t\ (dcheck\ t)$
$dcheck = dwrap\ check$
$check\ (TArr\ t_1\ t_2) = check'\ t_1 \gg check'\ t_2$
$check\ (TAll\ \_\ \_)$   $= \textbf{fail}$ "not a mono type"
$check\ \_$               $= \textbf{return}\ ()$
$check'\ (t,\_)$          $= dcheck\ t$
```

Figure 8.9: FPH monadic Haskell premises

```
$dwrap :: \forall\alpha\ .\ Deferrable\ \alpha \Rightarrow (\alpha \to I\ \alpha) \to \alpha \to I\ \alpha$
$dwrap\ f\ t = \textbf{do}\ (t_{out}, ()) \leftarrow \textbf{defer}\ (\lambda t_{in}\ \_ \to$
                   $\textbf{do unifOne}\ t\ t_{in}$
                      $t' \leftarrow f\ t$
                      $\textbf{unif}\ t_{out}\ t'$
                 $\textbf{return}\ t_{out}$
```

Figure 8.10: Example of evaluation control abstraction.

Functional Logic Programming. The essential problem with non-deterministic aspects is that the function to resolve it needs to make decisions, but is unable to do so based on what is known about the inputs at that point. Therefore, the idea is to delay execution until we know more about the output, and let expected output play a role in the decision process. Therefore, at specific places, we turn functions into relations, which has a close resemblance to Functional Logic Programming [Hanus, 1994].

With FLP, non-deterministic functions can be written as normal functions. The possible alternatives that these functions can take depends not only on the inputs, but also on scrutinizing the result. With evaluation strategies such as narrowing [Antoy, 1997], the search space is explored in a demand-driven way. Knowledge of context is pushed inwards, reducing possible alternatives, and causing evaluation to occur that refines the context even more.

With the defer and commit, we offer a poor man's mechanism to FLP. The delaying of choice and the scrutinizing of the choice is explicit. A commit is required to reduce the choice to at most one possibility. Yet, we have good reasons not to support the full generality of FLP. We want to integrate the inferencers specified in our language into mainstream compilers. Our approach makes only little demands on infrastructure. If it can cope with Algorithm W [Milner, 1978], then the proposed mechanisms of this chapter fit. Furthermore, we want to be able to use constraint solvers in some foreign language, or arbitrary Haskell libraries in our inference rules. This gives rise to problems with narrowing.

A difference with respect to FLP is our fixation and inspection of guesses. Consider an expression like *const x y*. For such an expression, the type of *y* is irrelevant and will not be scrutinized. However, we have several reasons to do so. We produce derivations, so we need a derivation of *y*. Decisions about the inference of *y* need to be made, even when the context does not make strong demands about which one. To make such a decision, we need to inspect which values are still guesses. As a consequence, more type information may be discovered, or even type errors that would otherwise go unnoticed, or which only arise much later (say, when generating code). Also, to deal with rules such as generalization properly, we need to know the difference between an unresolved guess and some fixed but unconstrained information. Furthermore, unresolved guesses retain memory which causes severe memory problems when growing unchecked in mainstream compilers, and when integrating with foreign code, we need invariants about which parts of values are resolved. Finally, examples such as in Figure 8.5 really require the guessing to be explicit and first class.

Logic Programming In a similar way as with FLP, our approach has strong ties with Logic Programming. One particular difference is that we disallow backtracking and of all possible rules demands that only one rule can succeed. Again, the reasons are related to efficiency and integration. However, there is an even more important reason: to be able to produce sensible type errors, and to prevent infinite searches in the presence of type errors.

Inferencer Frameworks. There are inferencer frameworks such as *HM* (X) [Sulzmann and Stuckey, 2008], which is based on a fixed set of type rules parametrized over some relation X, for which an inference algorithm needs to be given to obtain an inferencer for the full language. Such a framework is in fact orthogonal to Ruler, and Ruler can be used to construct the algorithm for X. In fact, the precise relation of Ruler to other constraint-based inference frameworks is that a Ruler specification can be seen as both a specification of constraints, annotated with the algorithm to solve them.

Type rule tools. The tool OTT [Sewell et al., 2007] generates code for proof assistants. SASyLF [Aldrich et al., 2008a] is such a proof assistant, and is tailored to proving properties about type systems, as is Twelf [Harper and Licata, 2007].

Tinkertype [Levin and Pierce, 2003] is a system that can also generate inference algorithms from type rules. However, the inference algorithms are not derived from the type rules. Instead, it depends on a repository with code for each relation to compose the inferencer.

8.4 Type Inferencer Syntax

8.4.1 Core Syntax

The syntax of the type inferencer language (named RulerCore) is given in Figure 8.11. A type inferencer is a triple $(\Sigma^*, r^*, H_\lambda)$ of schemes Σ^*, rules r^*, and some Haskell support code in the form of data-type declarations, some instances for them, and utility functions. A scheme Σ represents a function named s with inputs declared by environment Γ_{in}, and outputs by environment Γ_{out}. A scheme can be parametrized over some types α, which provides for a limited form of polymorphism for the inferencer rules. The inferencer rules in r^* with scheme name s define the function s. Each rule r consists of a (possibly empty) sequence of premises (c), and a conclusion (c_s).

It is important to realize that we are not defining type rules here. Schemes are not arbitrary relations, but are functions. The premises are statements, not predicates. Also, the order of the premises matter. Values for all inputs need to be available before a scheme can be instantiated. The rules and statements have a certain operational behavior. A rule evaluates successfully if and only evaluation of all its premises succeeds, and for each statement we give a brief description below. We make this more precise later.

The conclusion r_s of an inferencer rule defines to what names the actual parameters of the scheme s are bound in the context of the rule, and which local results are the outputs of the scheme. Expressed with bindings Δ_{in} and Δ_{out} respectively, where bindings are a mapping from formal names to the actual name.

Statements for premises come in different forms. There are a couple of statements that allow algorithms to be described:

- Evaluation of statement c_s instantiates a scheme, which means applying the scheme function to the inputs, and obtaining the outputs if this application is successful. For the input values, the values bound to the actual names are taken for the formal names (specified by bindings Δ_{in}). Similarly, the values bound to the formal name of the scheme are made available under the actual name (specified by bindings Δ_{out}).

- The unification statement attempts to unify the values bound to the names n_1 and n_2. A unification algorithm based on structural equality needs to be available for the values to which these two types are bound. Successful unification results in a substitution that makes the two values equal. This substitution is implicit and can be assumed to be applied everywhere.

- The execution statement allows monadic Haskell code to be executed. This code is a function taking the values bound to names n^* as parameter and returns a monadic value containing values for outputs m^*. We consider this expression language in more detail later. The purpose of these execution statements is to perform the actual computations needed to produce the values for the inputs of a scheme instantiation, and to inspect the outputs of it by means of pattern matching.
- The fixpoint statement repeatedly instantiates s, as long as the values bound to identifiers n^* change. The bindings Δ specify how the outputs are mapped back to the inputs after each iteration.

The statements that allow us to algorithmically deal with non-deterministic aspects:

- The defer statement represents one of the sequence of statements c_i^*, except that evaluation of it takes place at a later time. In the meantime, a guess (encoded as a fresh variable) for the output n is produced, and bottom-values for outputs n^*. For each guessable data type, we require that we can encode a variable as a value of this data type (denoted as $[\![v]\!]$). These guesses can be passed around as normal values.
- A commit statement refines a guess bound to v with the value bound to identifier n, and runs deferred statements, which may lead to other refinements of guesses.
- The fixate statement executes statements c^*. All guesses of type τ that were not committed during this execution are resolved by executing the deferred statements. Any remaining guesses are marked as fixed. These now represent opaque values that cannot be refined anymore.

Expressions in an exec-statement are Haskell functions in monad I that get the inputs passed as arguments and are obliged to return a product with the results. Hence the type of an expression e:

$$e :: \tau_{n_1} \to \ldots \tau_{n_k} \to I(\tau_{m_1}, \ldots, \tau_{m_l})$$

The I monad contains a hidden state, and support failure. In particular, the following operations are available:

$$\begin{array}{ll}
\textbf{commit} & :: \mathit{Deferrable}\ \alpha \Rightarrow \alpha \to \alpha \to I\ () \\
\textbf{defer} & :: (\mathit{Deferrable}\ \alpha, \mathit{Prod}\ \beta) \Rightarrow (\alpha \to I\ \beta) \to I\ (\alpha, \beta) \\
\textbf{unif} & :: \mathit{Unifyable}\ \alpha \Rightarrow \alpha \to \alpha \to I\ () \\
\textbf{unifOne} & :: \mathit{Unifyable}\ \alpha \Rightarrow \alpha \to \alpha \to I\ () \\
\textbf{update} & :: \mathit{Container}\ \alpha \Rightarrow \alpha \to I\ \alpha \\
\textbf{fail} & :: \mathit{String} \to I\ ()
\end{array}$$

We create a deferrable computation with *Defer*. It takes a monadic function that is only executed when a commit is performed on alpha. This monadic function produces the values for

$$
\begin{array}{rcll}
\Sigma^* & = & \Sigma_1,\ldots,\Sigma_k & \text{(schemes)} \\
\Sigma & = & \forall\overline{\alpha}.\ \Gamma_{\text{in}} \vdash_s \Gamma_{\text{out}} & \text{(scheme)} \\
r^* & = & r_{s_1},\ldots,r_{s_k} & \text{(rules)} \\
r_s & = & c^*\ ;\ c_s & \text{(rule)} \\
c^* & = & c_1,\ldots,c_k & \text{(statements)} \\
c & = & c_s & \text{(instantiate)} \\
 & | & n_1 \equiv n_2 & \text{(unification)} \\
 & | & \mathsf{exec}\, e :: n^* \to m^* & \text{(execution)} \\
 & | & \mathsf{fixpoint}^{n^*}_{\Delta}\ c_s & \text{(fixpoint)} \\
 & | & \mathsf{defer}^{m^*}_{n}\ c^*_1,\ldots,c^*_k & \text{(defer)} \\
 & | & \mathsf{commit}_{v_\tau}\ n & \text{(commit)} \\
 & | & \mathsf{fixate}_{\tau}\, c^* & \text{(fixate)} \\
c_s & = & \Delta_{\text{in}} \vdash_s \Delta_{\text{out}} & \text{(scheme instance)} \\
\Gamma & = & n_1 ::_\rho \tau_1,\ldots,n_k ::_\rho \tau_k & \text{(environment)} \\
\Delta & = & n_1 \mapsto m_1,\ldots,n_k \mapsto m_k & \text{(bindings)} \\
\rho & = & \text{d} & \text{(deferrable)} \\
 & | & \text{u} & \text{(unifyable)} \\
 & | & \emptyset & \text{(none)}
\end{array}
$$

With scheme names s, identifiers n, m, and v, collection of identifiers n^* and m^*, Haskell types τ, and expressions e.

Figure 8.11: Syntax of RulerCore.

the product β. Until this actually happened, the contents of the product may not be touched. The **unif** operation enforces structural equality between values α. In case of **unifOne**, only structural equality on the heads of the values. Finally, **update** brings all guesses in α up to date, and **fail** causes the inference to fail with a type error.

8.4.2 Syntactic Sugar

The previous section gave the core syntax for the type inferencer. For practical and didactic purposes, the examples in the previous section where given in a somewhat more convenient syntax that can be translated to the core syntax.

First of all, we assume a series of notational conventions involving sequences (lists), environments (maps), or sets:

- A sequence of, for example, statements is denoted by $c^* = \overline{c_i} = c_1 \ldots c_k$, where i ($1 \leqslant i \leqslant k$) is some index, and k is left implicit as it is clear from the context, i.e. when i ranges also over some list.

- A list of identifiers is denoted by $n^* = \overline{n_i} = n_1,\ldots,n_k$.

- The empty map, empty list, or empty set is written as $\emptyset$.

$$
\begin{array}{rcll}
\Sigma^* & = & \Sigma_1, \ldots, \Sigma_k & \text{(schemes)} \\
\Sigma & = & \forall \overline{\alpha}.\ s : d_1, \ldots, d_k & \text{(scheme signature)} \\
d & = & \mathbf{kw} & \text{(keyword decl)} \\
 & | & n \lhd_\rho \tau & \text{(input decl)} \\
 & | & n \rhd_\rho \tau & \text{(output decl)} \\
r^* & = & r_{s_1}, \ldots, r_{s_k} & \text{(rules)} \\
r_s & = & \dfrac{c_1 \ldots c_k}{c_s} & \text{(rule)} \\
c & = & c_s & \text{(instantiate)} \\
 & | & m_{H_\lambda,1} \equiv m_{H_\lambda,2} & \text{(unification)} \\
 & | & \mathsf{let}\ p_{H_\lambda} = e_{H_\lambda} & \text{(pure)} \\
 & | & p_{H_\lambda} \leftarrow m_{H_\lambda} & \text{(monadic bind)} \\
 & | & m_{H_\lambda} & \text{(monadic exec)} \\
 & | & \mathsf{fixpoint}_\Delta^{n^*}\ c_s & \text{(fixpoint)} \\
 & | & \mathsf{defer}_n^{m^*}\ c_1^*, \ldots, c_k^* & \text{(defer)} \\
 & | & \mathsf{commit}_{v_\tau}\ e_{H_\lambda} & \text{(commit)} \\
 & | & \mathsf{fixate}_\tau\, c^* & \text{(fixate)} \\
 & | & \mathsf{force}\, m_{H_\lambda} & \text{(force)} \\
c_s & = & s : i_1, \ldots, i_k & \text{(scheme instance)} \\
i & = & \mathbf{kw} & \text{(keyword)} \\
 & | & p_{H_\lambda} & \text{(value deconstruction)} \\
 & | & e_{H_\lambda} & \text{(value construction)}
\end{array}
$$

With keywords **kw**, Haskell patterns p_{H_λ}, Haskell expressions e_{H_λ}, and monadic Haskell expressions m_{H_λ}.

Figure 8.12: Syntax of RulerBase.

The syntactic sugar is given in Figure 8.12. A scheme declaration Σ for a scheme named s, now consists of a sequence of either an input or output declaration, or a keyword. For example, the scheme declaration for a scheme tp_expr:

$$\mathsf{tp\_expr} : (\Gamma \lhd \mathsf{Env}) \vdash (e \lhd \mathsf{Expr}) : (\tau \rhd_{\mathsf{d}} \mathsf{Type})$$

defines two inputs Γ and e with types Env and Expr respectively, and an output named τ with type Type (and the deferrable-property). To instantiate this scheme, the statement has the form: $\mathsf{tp\_expr} : \ldots \vdash \ldots : \ldots$, where at the places of the dots there is a Haskell pattern for an output, and a pure Haskell expression for an input (the reverse for the conclusion statement). The identifiers of a pattern can be referenced by expressions of subsequent statements of a rule. In the examples of Section 8.2, we left out the scheme names, because it is clear from the context.

The essential differences between Figure 8.11 and Figure 8.12 are:

- The syntactic sugar allows for Haskell expressions and patterns at places where originally only identifiers were expected. This syntactic sugar is translated to execution-statements with the appropriate inputs and outputs. Pattern match failures are translated to fail-expressions. We will also assume that pure Haskell expressions are automatically lifted into a monadic expression when needed. Also, an identifier occurring at multiple input-locations is replaced with unique identifiers with the necessary *equiv*-statements added to the front of the statement sequence.

- No special rules about the structure of monadic functions. These are normal Haskell monadic expressions in some monad I, and the commit, unify, and update-operations are functions that act in this monad.

- For the core language, only one deferred statement-sequence is allowed to succeed when triggered to evaluate. Here, we assume that more than one is allowed to succeed, but the first one from the left is taken.

- force is syntactic sugar for executing f m_{H_λ}. If the result is a guess, then a commit is done with a fresh value as parameter. If the result is still this fresh value, a commit is done with a fixed value. Otherwise, the result is ignored.

Identifiers occurring in expressions are brought up-to-date with respect to guesses just before evaluating the expressions.

Finally, we assume a number of pattern declarations, of which an example is given in Figure 8.13. These define special syntax for Haskell expressions (indicated with **input**) and patterns (indicated with **output**) for certain specific types, and the translation to Haskell. To disambiguate, the types of identifiers play a role. The pattern x can stand for the identifier x (say, if the type is *String*), or for the expression *EVar x* if type is *Expr*.

8.5 Operational Semantics

In this section we give a big-step operational semantics of the inferencer language introduced in Section 8.4. We first discuss some notation, then explain the evaluation rules. Evaluation

```
pattern Map String Ty where
  $x :: \tau, \Gamma$         input   insert $x$ $\tau$ $\Gamma$
pattern Expr where
  $x$                      output EVar $x$
  $f\ a$                   output EApp $f$ $a$
  $\lambda x\ .\ e$             output ELam $x$ $e$
pattern Ty where
  $[\alpha := \tau_1]\ \tau_2$  output singleSubst $\alpha$ $\tau_1 \mapsto ty$
pattern Ty where
  $\boxed{t}_b$                input   $(t, b)$
  $\boxed{t}$                  input   $(t, BYes)$
  $t$                      input   $(t, BNo)$
```

Figure 8.13: Examples of pattern declarations.

of the inferencer rules involves data manipulation. Some demands are made about the data in question. In particular, we require structural equality to be defined for data types. We finish this section with a discussion of these demands.

8.5.1 Notation

For the operational semantics, heaps $\mathcal{H}$, substitutions θ and derivations π are used for bookkeeping. Their syntax is given in Figure 8.14. Heaps are a mapping of locations (in our case, plain identifiers) to Haskell values. Substitutions keep track of information about guesses (identified by a variable v). Either a guess is resolved and represents some concrete value w of type τ, or will be resolved through a commit on another variable and is for the moment mapped to $\bot$, or represents a closure of the deferred statements. In the latter case, we store in a heap $\mathcal{H}$ entries for each identifier referenced by the deferred statements, store a scope identifier ζ representing the deepest scope in which the deferred statement is introduced (encoded as a number equal to the nesting-depth), and a rule identifier r. Each defer-statement introduces a unique rule identifier, which is a placeholder for a derivation. Derivations represents a partial derivation in an abstract way. The conclusions of each rule make up the nodes of the derivation-tree (with the values of their instantiation in heap $\mathcal{H}$). Statements that cannot be represented by this are represented with an opaque-leaf $\diamond$. These derivations may be partial and refer to sub-derivations named ι with $!\iota$.

Substitutions satisfy the usual substitution properties. Juxtaposition of substitutions $\theta_1 \theta_2$ represents the left-biased union of the two substitutions, with θ_1 applied to all entries q of θ_2. Applying a substitution θ_1 to a value w, denoted with θw, replaces each guess $[\![v]\!]$ with either w if $v \mapsto w_\tau \in \theta$ or itself otherwise. Application to a $\bot$-entry is the identity, and to a **deferred**-entry means applying it to the heap. Substitution application is lifted to environments, heaps, and derivations as well.

$$\begin{array}{lcll}
\mathcal{H} & = & n_1 := w_{1,\tau_1}, \ldots, n_k := w_{k,\tau_k} & \text{(heap)} \\
\theta & = & v_1 \mapsto q_1, \ldots, v_k \mapsto q_k & \text{(substitution)} \\
q & = & w_\tau & \text{(subst value)} \\
 & | & \bot & \text{(subst bot)} \\
 & | & \mathbf{deferred}^{\zeta,\iota}_{\mathcal{H}}\ c^*_1, \ldots, c^*_k & \text{(deferred)} \\
\pi & = & \dfrac{\pi}{\mathcal{H} \vdash c_s} & \text{(node)} \\
 & | & !\iota & \text{(reference)} \\
 & | & \diamond & \text{(opaque)} \\
 & | & \pi_1\ \pi_2 & \text{(and-cons)} \\
\Pi & = & \iota_1 \mapsto \pi_1, \ldots, \iota_k \mapsto \pi_k & \text{(named derivations)}
\end{array}$$

With Haskell value w, rule identifier ι, and scope identifier ζ.

Figure 8.14: Syntax of heaps, substitutions, and derivations.

We also use some notation concerning heaps and bindings. The lookup of a value for an identifier is written as $\mathcal{H}(n)$. With bindings we can take and rename entries in a heap: $\mathcal{H}(\Delta)$ is a heap which for each binding $n \mapsto n'$ has the value w_τ for n taken from $\mathcal{H}$ as n', i.e. $\mathcal{H}(\Delta)(n) = \mathcal{H}(n')$. We also use the reverse: $\Delta(\mathcal{H})(n') = \mathcal{H}(n)$. Juxtaposition of heaps stands for the left-biased union of the two.

8.5.2 Evaluation Rules

Overview. We can now give the evaluation rules of our big-step operational semantics. Figure 8.15 lists the structure of the evaluation rules. Given a statement c, the reduction relation gives a transition from a substitution θ_0 and heap $\mathcal{H}_0$ with values for the inputs of c, to an heap $\mathcal{H}_1$ containing values for the outputs of c and an updated substitution θ_1. The transition is labeled with a derivation π which can be considered a trace of the steps that were taken in order to make the transition. Similarly, Π contains (at least) a binding for each reference in derivation π and any reference of any derivation in Π itself. There are some variants of this reduction relation on the level of statements and rules. An important invariant is that the resulting heap is up to date with respect to the resulting substitution.

The semantics of substitution refinement by defaulting the guesses of a certain scope, starts with an initial substitution θ_0, and the current scope identifier ζ, and ends in a state θ_1. The purpose of this relation is to force the evaluation of deferred statements created in ζ of type τ, such that none of these remain in θ_1.

For the evaluation of (monadic) Haskell expressions, we construct an expression e and evaluate it in an execution environment H_λ, containing data type definitions, Haskell support code, augmented with bindings for inputs to the expression, including the substitution.

Given a type inferencer, a triple $(\Sigma^*, r^*, H_\lambda)$, and an instantiation of scheme Σ by means of statement c_s with a heap $\mathcal{H}_0$, evaluation of this statement with the inferencer rules is the transition

$$\emptyset\ ;\ \mathcal{H}_0\ ;\ 0\ ;\ \mathsf{fixate}_*\ c_s \ \rightarrow^{\Pi}_{\pi}\ \mathcal{H}_1\ ;\ \theta_1$$

$$\begin{array}{ll}
\theta_0 \;;\; \mathcal{H}_0 \;;\; \zeta \;;\; c \;\rightarrow^{\Pi}_{\pi}\; \mathcal{H}_1 \;;\; \theta_1 & \text{(statement reduction)} \\
\theta_0 \;;\; \mathcal{H}_0 \;;\; \zeta \;;\; c_1, \ldots, c_k \;\rightarrow^{\Pi}_{\pi}\; \mathcal{H}_1 \;;\; \theta_1 & \text{(statements reduction)} \\
\theta_0 \;;\; \mathcal{H}_0 \;;\; \zeta \;;\; r_s \;\rightarrow^{\Pi}_{\pi}\; \mathcal{H}_1 \;;\; \theta_1 & \text{(rule reduction)} \\
\tau \;;\; \theta_0 \;;\; \zeta \rightarrow^{\Pi}_{*} \theta_1 & \text{(scope defaulting)} \\
H_\lambda \vdash e_{H_\lambda} \rightarrow w & \text{(Haskell evaluation)}
\end{array}$$

Figure 8.15: Structure of the evaluation rules.

according to the smallest reduction relations satisfying the evaluation rules of Figure 8.16, Figure 8.17, Figure 8.18, and Figure 8.19. We explain these rules in more detail. Furthermore, we assume that the components of the inferencer-triple are available in the rules as a constant.

$$\frac{\begin{array}{cc} r_s \in r^* & \mathcal{H}_{\text{in}'} = \mathcal{H}_{\text{in}}(\Delta_{\text{in}}) \\ \mathcal{H}_{\text{out}} = \Delta_{\text{out}}(\mathcal{H}_{\text{out}'}) & \theta \;;\; \mathcal{H}_{\text{in}'} \;;\; \zeta \;;\; r_s \;\rightarrow^{\Pi}_{\pi}\; \mathcal{H}_{\text{out}'} \;;\; \theta' \end{array}}{\theta_1 \;;\; \mathcal{H}_{\text{in}} \;;\; \zeta \;;\; \Delta_{\text{in}} \vdash_s \Delta_{\text{out}} \;\rightarrow^{\Pi}_{\pi}\; \mathcal{H}_{\text{out}} \;;\; \theta'} \;\textsc{Scheme}$$

$$\frac{\theta' = \mathit{fst}\;(\mathit{run}\;(\mathbf{unif}\;\mathcal{H}(n_1)\;\mathcal{H}(n_2))\;\theta\;\zeta)}{\theta \;;\; \mathcal{H} \;;\; \zeta \;;\; n_1 \equiv n_2 \;\rightarrow^{\emptyset}_{\diamond}\; \emptyset \;;\; \theta'} \;\textsc{Unify}$$

$$\frac{\begin{array}{c} H_\lambda \vdash \mathit{run}\;(e\;\mathcal{H}(n_1)\ldots\mathcal{H}(n_k))\;\theta\;\zeta \rightarrow (\theta', (w_1, \ldots, w_l)) \\ \mathcal{H}' = m_1 \mapsto w_1, \ldots m_k \mapsto w_k \end{array}}{\theta \;;\; \mathcal{H} \;;\; \zeta \;;\; \mathsf{exece} :: n_1, \ldots, n_k \rightarrow m_1, \ldots, m_l \;\rightarrow^{\emptyset}_{\diamond}\; \mathcal{H}' \;;\; \theta'} \;\textsc{Exec}$$

$$\frac{\begin{array}{cc} r_s \in r^* & \theta_1 \;;\; \mathcal{H}_1 \;;\; \zeta \;;\; r_s \;\rightarrow^{\Pi}_{\pi}\; \mathcal{H}_2 \;;\; \theta_2 \\ \mathcal{H}_1(n^*) \neq \mathcal{H}_2(n^*) & \theta_2 \;;\; \mathcal{H}_2(\Delta)\;\mathcal{H}_2 \;;\; \zeta \;;\; \mathsf{fixpoint}^{n^*}_{\Delta}\; c_s \;\rightarrow^{\Pi}_{\pi'}\; \mathcal{H}_3 \;;\; \theta_3 \end{array}}{\theta_1 \;;\; \mathcal{H}_1 \;;\; \zeta \;;\; \mathsf{fixpoint}^{n^*}_{\Delta}\; c_s \;\rightarrow^{\Pi}_{\pi\,\pi'}\; \mathcal{H}_3 \;;\; \theta_3} \;\textsc{Fixstep}$$

$$\theta \;;\; \mathcal{H} \;;\; \zeta \;;\; \mathsf{fixpoint}^{n^*}_{\Delta}\; c_s \;\rightarrow^{\Pi}_{\pi}\; \mathcal{H}(\Delta)\;\mathcal{H} \;;\; \theta \;\textsc{Fixskip}$$

Figure 8.16: Evaluation rules for conventional statements.

Conventional statements. In Figure 8.17 are the rule for what we call the conventional statements. These are the statements in which type checking algorithms can be expressed. Type inference is not possible with these rules yet since this requires guessing.

The Scheme-rule represents instantiation of a scheme named s. An inferencer rule r_s is chosen and evaluated. The bindings dictate which values to take from the heap to use as inputs to the rule. Similarly, the bindings dictate under which name the outputs after evaluation are to be stored. These inferencer rules must be syntax directed. There should be only one r_s that can be applied.

For the unify-rule, the values bound to n_1 and n_2 are checked for equality with the *unify* function defined on the type of these values. When the types involve guesses, this may lead to discovery of more type information about guesses and an updated substitution.

In the exec-rule, the monadic code is executed with the values of $n_1, \ldots, n_k$ as parameter. The monadic code may update the substitution, or cause the statement to fail. If the execution succeeds, the returned product of the monadic code contains the values for in the output-heap.

Finally, with the fixpoint-rule a scheme-statement can be repeatedly executed as long as it causes one of the values of n^* to change. For each repetition, the bindings Δ dictate which outputs are the inputs of the next iteration. In this case there may be more than one applicable rule r_s. However, to make a step, evaluation of the inferencer rule must cause a change of value n.

$$\frac{\begin{array}{c}\theta_{\text{in}} \mathbin{;} \Delta_{\text{in}}(\mathcal{H}_{\text{in}}) \mathbin{;} \zeta \mathbin{;} c_1, \ldots, c_k \rightarrow^{\Pi}_{\pi_1,\ldots,\pi_k} \mathcal{H}' \mathbin{;} \theta_{\text{out}} \\ \pi = \dfrac{\pi_1 \ldots \pi_k}{\mathcal{H}' \vdash (\Delta_{\text{in}} \vdash_s \Delta_{\text{out}})} \qquad \mathcal{H}_{\text{out}} = \mathcal{H}'(\Delta_{\text{out}})\end{array}}{\theta_{\text{in}} \mathbin{;} \mathcal{H}_{\text{in}} \mathbin{;} \zeta \mathbin{;} c_1, \ldots, c_k ; \Delta_{\text{in}} \vdash_s \Delta_{\text{out}} \rightarrow^{\Pi}_{\pi} \mathcal{H}_{\text{out}} \mathbin{;} \theta_{\text{out}}} \; \textsc{Rule}$$

$$\frac{\theta_i \mathbin{;} (\theta_i \mathcal{H}_i \ldots \mathcal{H}_1) \mathbin{;} \zeta \mathbin{;} c_i \rightarrow^{\Pi}_{\pi_i} \mathcal{H}_{i+1} \mathbin{;} \theta_{i+1}, \; 1 \leqslant i \leqslant k}{\theta_1 \mathbin{;} \mathcal{H}_1 \mathbin{;} \zeta \mathbin{;} c_1, \ldots, c_k \rightarrow^{\Pi}_{\pi_1 \ldots \pi_k} \theta_{k+1} \mathcal{H}_{k+1} \ldots \mathcal{H}_1 \mathbin{;} \theta_{k+1}} \; \textsc{Statements}$$

Figure 8.17: The apply rules.

In Figure 8.17 are evaluation rules for a chosen inferencer rule r_s. The bindings of the conclusion specifies under what names the inputs to the rule need to be presented to the statements. Likewise, the bindings also specify under what names the values of the outputs of the rule are available. Successful evaluation of a rule means that a derivation π has been produced, of which the current rule forms the root, and the derivations of the premises are its immediate children.

Evaluation of a sequence of statements causes the heap to accumulate the outputs of the statements already executed so far. The outputs of predecessors of a statement in this sequence are also available as input to the statement. Since each evaluation of a statement potentially causes more information to be known about guesses in such outputs, the most recent substitution is applied to these predecessor-outputs first.

Non-deterministic statements To deal with guesses, there are the statements that deal with non-determinism, which we will call the non-deterministic statements. Their semantics is made precise in Figure 8.18.

For a Defer-statement, guesses are produced as outputs for $n, m_1, \ldots, m_l$. A closure for the statements $c_1, \ldots, c_k$ is stored as substitution for the guess of n as closure. A commit on this guess leads to the execution of these statements, and to the production of values for the

$$\frac{\begin{array}{c} v, v_1, \ldots, v_l, \iota \text{ fresh} \qquad \mathcal{H} = n \mapsto [\![v]\!], m_1 \mapsto [\![v_1]\!], \ldots, m_l \mapsto [\![v_l]\!] \qquad \mathcal{H}_{\text{out}} = \mathcal{H}\mathcal{H}_{\text{in}} \\ \theta_1 = v_1 \mapsto \bot, \ldots, v_l \mapsto \bot \qquad \theta_2 = v \mapsto \mathbf{deferred}^{\zeta,\iota}_{\mathcal{H}_{\text{out}}}\ c_1^*, \ldots, c_k^* \end{array}}{\theta \,;\, \mathcal{H}_{\text{in}} \,;\, \zeta \,;\, \mathsf{defer}^{m_1,\ldots,m_l}_n\ c_1^*, \ldots, c_k^* \ \rightarrow^{\emptyset}_{!\iota}\ \mathcal{H}_{\text{out}} \,;\, \theta_1\theta_2\theta} \ \textsc{Defer}$$

$$\frac{\begin{array}{c} v \mapsto \mathbf{deferred}^{\zeta,\iota}_{\mathcal{H}}\ c^{**} \in \theta \qquad c_1, \ldots, c_k \in c^{**} \\ \theta_1 = v \mapsto \mathcal{H}_{\text{in}}(n), \theta_{\text{in}} \qquad \theta_1 \,;\, \theta_1\mathcal{H} \,;\, \zeta \,;\, c_1, \ldots, c_k \ \rightarrow^{\Pi}_{\pi}\ \mathcal{H}' \,;\, \theta_2 \\ \theta_{\text{out}} = \left\{ v \mapsto w \mid n \mapsto [\![v]\!] \in \mathcal{H}, n \mapsto w \in \mathcal{H}' \right\}, \theta_2 \qquad \iota \mapsto \pi \in \Pi \end{array}}{\theta_{\text{in}} \,;\, \mathcal{H}_{\text{in}} \,;\, \zeta \,;\, \mathsf{commit}_{v_\tau}\ n \ \rightarrow^{\Pi}_{\Diamond}\ \emptyset \,;\, \theta_{\text{out}}} \ \textsc{CommitVar}$$

$$\frac{\begin{array}{c} v \mapsto w \in \theta \\ n' \text{ fresh} \qquad \mathcal{H} = n' \mapsto w, n \mapsto \mathcal{H}_{\text{in}}(n) \qquad \theta_{\text{in}} \,;\, \mathcal{H} \,;\, \zeta \,;\, n' \equiv n \ \rightarrow^{\Pi}_{\Diamond}\ \mathcal{H}' \,;\, \theta_{\text{out}} \end{array}}{\theta_{\text{in}} \,;\, \mathcal{H}_{\text{in}} \,;\, \zeta \,;\, \mathsf{commit}_{v_\tau}\ n \ \rightarrow^{\Pi}_{\Diamond}\ \emptyset \,;\, \theta_{\text{out}}} \ \textsc{CommitVal}$$

$$\frac{\begin{array}{c} \theta_{\text{in}} \,;\, \mathcal{H}_{\text{in}} \,;\, \zeta \,;\, c^* \ \rightarrow^{\Pi}_{\pi}\ \mathcal{H}_{\text{out}} \,;\, \theta_1 \\ \tau \,;\, \theta_1 \,;\, \zeta \rightarrow^{\Pi}_{*} \theta_{\text{out}} \qquad \mathsf{deferred}(\zeta, \theta_{\text{out}}) = \emptyset \end{array}}{\theta_{\text{in}} \,;\, \mathcal{H}_{\text{in}} \,;\, \zeta - 1 \,;\, \mathsf{fixate}_{\tau}\, c^* \ \rightarrow^{\Pi}_{\pi}\ \theta_{\text{out}}\mathcal{H}_{\text{out}} \,;\, \theta_{\text{out}}} \ \textsc{Fixate}$$

Figure 8.18: Evaluation rules for non-determinism annotations.

guesses $m_1, \ldots, m_l$. Committing on these latter guesses is not possible, since the substitution for these guesses is mapped to $\bot$. This closure is introduced in scope ζ and contains a unique identifier ι which is the name of the derivation that is produced later. A reference to this derivation is returned as the derivation of the Defer-statement.

Evaluation of the Commit-statement leads to the evaluation of the deferred statements. The heap stored in the closure is updated to the current substitution, and the substitution reflects the newly found information about the guess. Then, one of the sequences of statements is chosen and evaluated. The evaluation has caused outputs to be produced for values that were before represented as a guess to a $\bot$-substitution. Subsequently, the substitution is updated such that these substitutions do not map to $\bot$ anymore, but to their newly produced value.

Finally, there is the Fixate-statement. Its statement is evaluated in a subscope ζ. After evaluation, the remaining guesses are defaulted such that no deferred statement is left for scope ζ. The rules for defaulting are specified in Figure 8.19.

A deferred guess is committed to with either a *flexible* guess as value, or with a *fixed* unknown value. In both cases, of the deferred statement-sequences must be able to evaluate with this newly found information. In the first case, such deferred statement observes a guess as value for its input, and is allowed to refine it. In the later case, the value may not be touched.

Since committing to a flexible guess results in the introduction of a guess in the scope, in order to end up with no guesses in the scope in the end, each commit to a flexible guess must

$$\frac{\begin{array}{c} n, v' \text{ fresh} \qquad v_\tau \in \mathrm{dom}(\theta_{\text{in}}) \qquad \mathcal{H} = n \mapsto [\![v']\!] \\ \theta = v' \mapsto \mathbf{deferred}^{\zeta,\iota}_{\emptyset}\ \{\emptyset\} \qquad \theta\,\theta_{\text{in}} \,;\, \mathcal{H} \,;\, \zeta \,;\, \mathsf{commit}_{v_\tau}\ n \rightarrow^{\Pi}_{\Diamond} \mathcal{H}' \,;\, \theta_{\text{out}} \end{array}}{\tau \,;\, \theta_{\text{in}} \,;\, \zeta \rightarrow^{\Pi}_{*} \theta_{\text{out}}}\ \textsc{Flex}$$

$$\frac{\begin{array}{c} n \text{ fresh} \qquad v_\tau \in \mathrm{dom}(\theta_{\text{in}}) \\ \mathcal{H} = n \mapsto [\![v]\!]_F \qquad \theta\,\theta_{\text{in}} \,;\, \mathcal{H} \,;\, \zeta \,;\, \mathsf{commit}_{v_\tau}\ n \rightarrow^{\Pi}_{\Diamond} \mathcal{H}' \,;\, \theta_{\text{out}} \end{array}}{\tau \,;\, \theta_{\text{in}} \,;\, \zeta \rightarrow^{\Pi}_{*} \theta_{\text{out}}}\ \textsc{Rigid}$$

$$\frac{\tau \,;\, \theta_{\text{in}} \,;\, \zeta \rightarrow^{\Pi}_{*} \theta \qquad \tau \,;\, \theta \,;\, \zeta \rightarrow^{\Pi}_{*} \theta_{\text{out}}}{\tau \,;\, \theta_{\text{in}} \,;\, \zeta \rightarrow^{\Pi}_{*} \theta_{\text{out}}}\ \textsc{Trans} \qquad\qquad \tau \,;\, \theta \,;\, \zeta \rightarrow^{\Pi}_{*} \theta\ \textsc{Finish}$$

Figure 8.19: Defaulting rules.

lead to refinements of guesses. All guesses that are essentially unconstrained will then end up with fixed unknowns.

$\textbf{type}\ I\ \alpha = \mathit{ErrorT}\ \mathit{Err}\ (\mathit{State}\ (\theta, \zeta))\ \alpha$
$\mathit{run} :: I\ \alpha \rightarrow \theta \rightarrow \zeta \rightarrow (\theta, \alpha)$

Figure 8.20: The *run* function.

Haskell semantics. We use Haskell to specify the monadic functions. The type of the monad is given in Figure 8.20. It is a conventional combination between the error monad and the state monad. The *run* function is the interface between the semantic world and the monad world. If the monadic evaluation is successful, the premise with *run* succeeds and there has been a state transition into the monad and back. If the evaluation results in an *Err*, the premise with *run* does not hold.

8.5.3 Data-type Semantics

The semantics of the previous section makes some assumptions about the types of identifiers occurring in de inferencer rules. This functionality needs to be available in terms of instances for the type classes listed in Figure 8.21. This functionality does not have to be available for all types. Most of this functionality can be generically derived from the structure of the types.

A *Container* instance is required to be defined for all types containing guesses. Substitution application $\theta \mapsto w$ replaces all occurrences of guess $[\![v]\!]$ with w', given $v \mapsto w'_\tau \in \theta$. It generically handles the substitution of guesses, and uses *appSubst* to traverse the type.

```
class Container α where
   appSubst :: (∀β . Container β ⇒ β → β) → α → α
class Unifyable α where
   unify :: (∀β . Unifyable β ⇒ β → β → I ())
            → α → α → I ()
class Deferrable α where
   deferVars        :: α → {GuessVar}
   mkDeferValue     :: GuessVar → α
   mkFixedValue     :: GuessVar → α
   matchDeferValue  :: α → Maybe GuessVar
   matchFixedValue  :: α → Maybe GuessVar
```

Figure 8.21: Semantics on data.

```
unif w₁ w₂ = unif′ unif w₁ w₂
unifOne w₁ w₂ = unif′ (\_ _ → return ()) w₁ w₂

unif′ r w₁ w₂ = do w₃ ← update w₁; w₄ ← update w₂
                   unif″ r w₁ w₂

unif″ _ w₁   w₂   | w₁ ≡ w₂          = return ()
unif″ _ ⟦v₁⟧ ⟦v₂⟧                    = compose v₁ v₂
unif″ _ ⟦v⟧  w    | v ∉ deferVars w  = commit v w
                  | otherwise        = fail "occur check"
unif″ _ w    ⟦v⟧  | v ∉ deferVars w  = commit v w
                  | otherwise        = fail "occur check"
unif″ r w₁   w₂                      = unify r w₁ w₂
```

Figure 8.22: Unification and guess specialization.

For the type of the identifiers of the unification rule a *Unifyable* instance needs to be defined, which asks for a definitely of the *unify* function. Its sole purpose is to succeed if and only if the heads of the two inputs are structurally equal. For the rest, it should delegate to its recursion parameter. This function does not have to deal with guesses, since those are handled generically by the **unif** function (Figure 8.22).

The **unif** function deals with guesses by committing concrete type information to the guess, unless both values are guesses. In that case, it takes the composition of the two. This means that the two guesses are substituted with a single guess, such that when information is committed to this single guess (in the minimal scope of the two), the deferred statements of the original guesses are sequenced after each other. Since we require confluence with respect to the order information about guesses is found, the execution order of these guesses is allowed to be arbitrary. Also note that not all *Unifyable* types have to be *Deferrable*, depending on the properties of the type. We omitted this detail here, as it is only a minor detail, and the code for **unif** would be considerably more complicated.

8.6 Conclusion

Type rules of declarative type systems contain non-deterministic aspects. These aspects are problematic when writing an inference algorithm. We presented a domain specific language for inferencers that has special syntax to formulate algorithms to resolve these non-deterministic aspects. The main concept is a first-class guess, which acts as a remote control to a deferred derivation. By manipulating guesses, the deferred derivations can be scheduled such that decisions are made at the moment sufficient information has become available. The result is that we can write inference algorithms by means of annotating the declarative rules of a type system, describing the global scheduling locally, without breaking the relative isolation of the type rules, and without breaking soundness with respect to the original rules.

Future Work We intend to formalize the type system of UHC [Dijkstra et al., 2007a], and generate portions of the inference algorithm from this description. We made design decisions that the generated algorithm to be reasonably efficient. Although we conceptually explained semantics of the language in monadic terms, we actually generate code for multi-pass higher-order attribute grammars. An open question is still if we can keep the complexity of some of UHC's efficient data structures hidden from the type rules.

8.7 Static Semantics

In this section we define the static semantics of the type inference language. This semantics expresses when a type inferencer $(\Sigma^*, r^*, H_\lambda)$ is correctly typed. Concretely, this means that all identifiers are defined before used, all used schemes are defined, identifiers participating in equality, defer and commit statements have the required properties defined for their types, and Haskell fragments have a type corresponding to the type of its inputs and outputs. When this is the case, then compiling the inferencer rules to an algorithm in Haskell gives a type correct Haskell program, and during the evaluation of the inferencer rules according to the operational semantics of Section 8.5, the values in the heap have the type of the identifiers if this was the case for the initial heap.

$$\begin{array}{ll} \vdash r_s & \text{(rule judgment)} \\ \overline{\alpha}\,;\Gamma_{\text{in}} \vdash c : \Gamma_{\text{out}} & \text{(statement judgment)} \\ H_\lambda \vdash e : \tau & \text{(Haskell judgment)} \end{array}$$

Figure 8.23: Structure of static semantics judgments.

The structure of the typing judgments is given in Figure 8.23. The rules may refer to the set of schemes Σ^*, which is assumed as a constant. All schemes are explicitly typed. Local identifiers of an inferencer rule have implicit types which are directly related to the explicit types of schemes due to bindings, or to a type of a Haskell fragment due to inputs and outputs.

We give type rules for two typing judgments, one for a rule and one for a statement. The other judgments are rather trivial and left out. For the validity of schemes we want to remark that the names of identifiers in the input environment must be disjoined to those of the output environment, and that all types in the schemes must have a correct kind with respect to the types in H_λ. About the typing judgment for statements, we remark that it mentions types $\overline{\alpha}$. These are the types over which the scheme, rule, and statements are polymorphic. Technically, the types of identifiers in the environments may have free variables, but only if their explicitly occur in $\overline{\alpha}$. Also, the output environment contains exactly the types for the outputs of the premise, whereas the input environment contains at least the types for the identifiers that are input to the premise.

$$\frac{\begin{array}{c}\forall\overline{\alpha}.\,\Gamma_{\text{in}} \vdash_s \Gamma_{\text{out}} \in \Sigma^*(s) \qquad \text{dom}(\Gamma_{\text{in}}) = \text{dom}(\Delta_{\text{in}}) \qquad \text{dom}(\Gamma_{\text{out}}) = \text{dom}(\Delta_{\text{out}}) \\ \overline{\alpha}\,;\Gamma_{\text{in}}(\Delta_{\text{in}})\,\Gamma'_{j<i} \vdash c_i : \Gamma'_i,\ 1 \leqslant i \leqslant k \qquad \Gamma_{\text{out}}(\Delta_{\text{out}}) \subseteq \Gamma_{\text{in}}(\Delta_{\text{in}})\,\Gamma'_{j\leqslant k}\end{array}}{\vdash c_1,\ldots,c_k;(\Delta_{\text{in}} \vdash_s \Delta_{\text{out}})}\ \textsc{Rule}$$

Figure 8.24: Rule typing rule.

To type check an inferencer rule, we check that a scheme has been defined for it, and verify the define-before-use requirement on the premises. Outputs of these premises are accumulated, and the next premise in the sequence may use any of these outputs. The local identifiers that are connected to the types of the scheme due to bindings, must have types that agree with the types of the scheme.

The typing rules for statements are listed in Figure 8.25. The typing judgement states that given some types $\overline{\alpha}$, and an environment Γ_{in} stating which identifiers are in scope and what type and properties these have, that the statement produces outputs with types Γ_{out}. We now focus at some aspects of these rules.

$$\frac{\forall\overline{\beta}.\,\Gamma'_{\mathrm{in}} \vdash_s \Gamma'_{\mathrm{out}} \in \Sigma^*(s) \qquad [\overline{\beta} := \overline{\tau}]\,\Gamma'_{\mathrm{in}} \subseteq \Gamma_{\mathrm{in}}(\Delta_{\mathrm{in}})}{\overline{\alpha}\,;\Gamma_{\mathrm{in}} \vdash (\Delta_{\mathrm{in}} \vdash \Delta_{\mathrm{out}}) : \Delta_{\mathrm{out}}([\overline{\beta} := \overline{\tau}]\,\Gamma'_{\mathrm{out}})}\ \textsc{Scheme}$$

$$\frac{n_1 ::_{\rho_1} \tau \in \Gamma \qquad n_2 ::_{\rho_2} \tau \in \Gamma \qquad \rho_1 \neq \emptyset \qquad \rho_2 \neq \emptyset}{\overline{\alpha}\,;\Gamma \vdash n_1 \equiv n_2 : \emptyset}\ \textsc{Unify}$$

$$\frac{\begin{array}{c}\tau_{\mathrm{in}} = \Gamma_{\mathrm{in}}(n_1) \to \ldots \to \Gamma_{\mathrm{in}}(n_k) \qquad \tau_{\mathrm{out}} = (\tau_{1,\rho_1}, \ldots, \tau_{l,\rho_l}) \\ H_\lambda \vdash e : \forall\overline{\alpha}.\ \tau_{\mathrm{in}} \to I\ \tau_{\mathrm{out}} \qquad \Gamma_{\mathrm{out}} = m_1 ::_{\rho_1} \tau_1, \ldots, m_l ::_{\rho_l} \tau_l\end{array}}{\overline{\alpha}\,;\Gamma_{\mathrm{in}} \vdash \mathsf{exec}\,e :: (n_1 \ldots n_k) \to (m_1 \ldots m_l) : \Gamma_{\mathrm{out}}}\ \textsc{Execution}$$

$$\frac{\begin{array}{c}\overline{\alpha}\,;\Gamma_{\mathrm{in}} \vdash c_s : \Gamma_{\mathrm{out}} \\ \Gamma_{\mathrm{in}}(m_i) = \Gamma_{\mathrm{out}}(m'_i),\ 1 \leqslant i \leqslant k \qquad n^* \subseteq \mathrm{dom}(\Gamma_{\mathrm{in}}) \qquad n^* \subseteq \mathrm{dom}(\Gamma_{\mathrm{out}})\end{array}}{\overline{\alpha}\,;\Gamma_{\mathrm{in}} \vdash \mathsf{fixpoint}^{n^*}_{m_1 \mapsto m'_1, \ldots, m_k \mapsto m'_k}\ c_s : \Gamma_{\mathrm{out}}}\ \textsc{Fixpoint}$$

$$\frac{\Gamma_{\mathrm{in}}\,\Gamma'_{j<i} \vdash c_i : \Gamma'_i,\ 1 \leqslant i \leqslant k \qquad \Gamma_{\mathrm{out}} = \Gamma'_{j \leqslant k}\ \{n, m^*\} \qquad n ::_{\mathrm{d}} \tau \in \Gamma_{\mathrm{out}}}{\overline{\alpha}\,;\Gamma_{\mathrm{in}} \vdash \mathsf{defer}^{m^*}_{n}\ c^*_1, \ldots, c^*_k : \Gamma_{\mathrm{out}}}\ \textsc{Defer}$$

$$\frac{n ::_{\mathrm{d}} \tau \in \Gamma_{\mathrm{in}}}{\overline{\alpha}\,;\Gamma_{\mathrm{in}} \vdash \mathsf{commit}_{v_\tau}\ n : \emptyset}\ \textsc{Commit} \qquad \frac{\overline{\alpha}\,; Env'_{j<i}, \Gamma_{\mathrm{in}} \vdash c_i : \Gamma'_i,\ 1 \leqslant i \leqslant k}{\overline{\alpha}\,;\Gamma_{\mathrm{in}} \vdash \mathsf{fixate}_\tau\, c_1, \ldots, c_k : Env'_{j \leqslant k}}\ \textsc{Fixate}$$

Figure 8.25: Statement typing rule.

Polymorphism. A limited form of polymorphism is allowed for the types of the inferencer rules, by allowing the types to be parametrized over type variables $\overline{\alpha}$. This is useful when in order to be able to reuse some of the inferencer rules, or when the syntax of the language we are writing an inferencer for is itself polymorphic (for example, parametrized over the types of variables). In fact, we will silently also allow ad-hoc polymorphism by having a set of type class constraints over these variables $\overline{\alpha}$, for example to be able to show values of such a polymorphic type, to test for equality, or to use such values in maps.

This polymorphism is visible at two places. In the scheme-rule, when instantiating a scheme polymorphic in $\overline{\beta}$, we can choose the types for these variables. And in the execution-rule, the type of the monadic function must be polymorphic over the variables the scheme itself is polymorphic.

Haskell. To type monadic functions, we use Haskell's typing relation with an initial environment H_λ (containing several utility functions, data types, etc.). The monadic function is a function of taking some of the inputs of the execution-statement, and returning the outputs in monad I.

Properties. Some statements can require additional semantics defines on the values they operate on. For example, in order to test two values for equality in the unify-rule, we require a *Unify*-instance to be defined on the type. This is encoded in the language as a property ρ of an identifier. There are three properties: none, unifyable, and deferrable. When an identifier has the deferrable property, there are instances of both *Unifyable* as *Deferrable* for its type. The commit and defer statements require this deferrable property to be defined for the identifiers they act on.

8.8 Soundness Almost For Free

In general, it is hard to prove that a concrete type inference algorithm is sound with respect to the type system it is based on. Formally this means that if the inferencer manages to infer a type for some program value, that this is indeed a correct type for the program value according to the type system. However, when the inference algorithm is described with the inferencer rules, we almost get the soundness almost for free.

Almost free. The almost-part is due to some assumptions that we need to make:

- The inferencer rules contains concrete algorithms in the form of Haskell fragments at the places where type rules has (non-judgement) premises. For example, when a type rule has a premise $\overline{\alpha} \mathrel{\#} \Gamma$, the the type inferencer rule has a premise

  $$\textbf{let}\ \overline{\alpha} = \mathit{ftv}\ \mathit{ty} - \mathit{ftv}\ \Gamma$$

  . In the first case, we only state a demand on $\overline{\alpha}$, in the later case we precisely define what $\overline{\alpha}$ is. For soundness, we need the guarantee that the code fragment ensures that the constraint on $\overline{\alpha}$ is satisfied.

- The inferencer rules may be more fine-grained than the type rules in order to deal with syntax-directness and explicit scheduling or pipelining of certain rules. For example, for the HML type system, we have a master rule for instantiation with the sole purpose to orchestrate the specific rules for instantiation. Therefore, we require an erasure function $\lfloor \cdot \rfloor :: \pi \rightarrow \pi$ that eliminates the extra structure from the derivation, such that a derivation is left that matches the structure of the type rules.

Notation. Let $D_T(s, \mathcal{H})$ stand for the set of all derivations π that are valid according to type system T for an instantiation of scheme s of T, with the bindings for the identifiers of the scheme in $\mathcal{H}$.

Let $\Pi(\pi)$ stand for the substitution and merging of a derivations reference $!\iota$ in π with the derivations named ι in Π. Let $\Pi_*(\pi)$ stand for the fixpoint. If Π is complete, then there is no reference left in the result.

Soundness.

Theorem 8.8.1. Suppose that $(\Sigma^*, r^*, H_\lambda)$ is an inferencer for some type system T, $\mathcal{H}_0$ and $\mathcal{H}_1$ are some heap, and c_s is an instantiation of one of the schemes of T. Now suppose that there is some substitution θ such that:

$$\emptyset \,;\, \mathcal{H}_{\text{in}} \,;\, 0 \,;\, \mathsf{fixate}\, c_s \;\rightarrow^{\Pi}_{\pi}\; \mathcal{H}_{\text{out}} \,;\, \theta$$

Then:

$$\lfloor \Pi_*(\pi) \rfloor \in D_T(s, \mathcal{H}_{\text{out}}\mathcal{H}_{\text{in}})$$

Proof. We give a sketch. First note that only fixate-statements introduce a scope, and also guarantee that there are no deferred statements remaining in this scope. More concretely, θ does not have any deferred statements. Derivations references are only introduced by a deferred-statement, which also brings equally named deferred statements in scope. Since these have all been resolved, it means that Π is complete, and thus $\Pi_*(\pi)$ is a full derivation without any references.

By taking the erasure of this derivation, we obtain a derivation π'. In order to show that $\pi' \in D_T(s, \mathcal{H}_{\text{out}}\mathcal{H}_{\text{in}})$, we need to prove for each node in π', the bindings in its heap $\mathcal{H}$, satisfy the premises of the rule in T corresponding with the node.

If there is no guessing involved, then we know that Haskell fragments have successfully run and ensured that these premises did hold. Now, when guessing was involved, this means that a certain order of evaluation was taken in order to produce the values. Some values where inspected before the full values where known. We now need to show that the same values would be observed when this evaluation would have occurred at the very end of the inference process.

We use an important property: all values are constant up to the guesses. Once a commit has been done on a guess, it is never rolled back. This has as consequence that once we observe that a value has a certain structure, this value can be considered to always keep having this structure. Therefore, a Haskell fragment executing too late does not matter, but what if it executed too early, and thus saw only a partial value?

There are two cases to consider. The first case is that enough of the value was known to complete evaluation. These portions of the value cannot have changed until the end of the inference process, and thus that evaluation would still be valid at a later time. In the second case, not enough of the value was known, and some of the evaluation was deferred. In that case, since all deferred statements have executed, this means that the evaluation was done successfully at a later time. □

Remarks. This section talked about soundness only. Dual to soundness is completeness. Unfortunately, completeness cannot be proved in general, because we can encode type systems for which no inference algorithm exists (for example, implicitly typed System F). However, we can ask ourselves the question what constraints we need on the type system and the Haskell fragments in the inferencer rules, in order to be able to prove completeness. As for now, we do not have an answer to this, otherwise, interesting question. Also, we note that one wants to experiment with inference algorithms. Soundness with respect to the type system is immediately wanted, but completeness is something we expect only to achieve after playing around sufficiently and making the right implementation decisions.

8.9 Future Work

Limitations. The inference algorithm has as property that refinements on a guess are never undone. This eliminates the need for backtracking, which ensures that a fixed traversal over the AST is sufficient to construct the derivation. We believe that this works for many kinds of non-deterministic aspects of type rules. On the other hand, there are type rules that cannot be mapped to a fixed traversal, but require a constraint-solving algorithm (for example, Constraint Handling Rules). For example, type rules related to overloading in Haskell. One particular question is if we can discover what non-deterministic aspect cannot be dealt with by a single traversal, and map all rules that depend on it to CHR constraints.

Extensions

- Higher-order abstractions for type rules.
- Transformations of certain patterns. Automatic insertion of *fresh* when input is not available yet. Automatic insertion of $\equiv$ when putting an input at the place of an output

From interpreter to compiler. The practical intention of our research is to generate the type inferencer code for the UHC. This requires compiling the inferencer rules instead of interpreting them. Declarative aspects of type rules prevent a straightforward mapping to attribute grammar code. The ideas that we presented in this chapter can serve as a basis for an inference systems based on attribute grammars.

Efficiency. We imposed some constraints, such as syntax-directed rules for the inference, and disallowing backtracking, in order to have a system that can be implemented in terms of proved conventional and reasonably efficient technology. This, however, only applies to the evaluation process and scheduling of the rules. The actual computations are performed by Haskell code, referenced by the rules, that manipulates the data structures such as environments and types.

Experience with the UHC shows that it is the efficiency of this code that matters in order to scale up to typing real-world programs. For example, UHC uses a more complicated data structure to represent an environment than a conventional map, in order to make some operations cheaper and overall memory consumption lower. As a result, some extra invariants need to be maintained which make the use of this data structure more complicated. Since the inferencer rules are a higher-level abstraction, we expect that the conventional interface can be provided to the rules, while we ensure that the extra work required for these special data structures is handled by inserting special annotations at the appropriate places.

9 AGs with Dependent Types

Attribute Grammars (AGs) are a domain-specific language for functional and composable descriptions of tree traversals. Given such a description, it is not immediately clear how to state and prove properties of AGs formally. To meet this challenge, we apply dependent types to AGs. In a dependently typed AG, the type of an attribute may refer to values of attributes. The type of an attribute is an invariant, the value of an attribute a proof for that invariant. Additionally, when an AG is cycle-free, the composition of the attributes is logically consistent. We present a lightweight approach using a preprocessor in combination with the dependently typed language Agda.

9.1 Introduction

Functional programming languages are known to be convenient languages for implementing a compiler [Appel, 1998]. As part of the compilation process, a compiler computes properties of Abstract Syntax Trees (ASTs), such as environments, types, error messages, and code. In functional programming, these syntax-directed computations are typically written as *catamorphisms*[1]. An *algebra* defines an inductive property in terms of each constructor of the AST, and a catamorphism applies the algebra to the AST. Catamorphisms thus play an important role in a functional implementation of a compiler.

Attribute Grammars (AGs) [Knuth, 1968] are a domain-specific language for *composable* descriptions of catamorphisms. AGs facilitate the description of complex catamorphisms that typically occur in complex compiler implementations.

An AG extends a context-free grammar by associating *attributes* with nonterminals. Functional *rules* are associated with productions, and define values for the attributes that occur in the nonterminals of associated productions. As AGs are typically embedded in a host language, the rules are terms in the host language, which may additionally refer to attributes. Attributes can easily be composed to form more complex properties. An AG can be compiled to an efficient functional algorithm that computes the synthesized attributes of the root of the AST, given the root's inherited attributes.

It is not immediately clear how to formally specify and write proofs about programs implemented with AGs. For example, it is common to prove that a type inferencer is a sound and complete implementation of a type system, and that the meaning of a well typed source program is preserved. *Dependent types* [Bove and Dybjer, 2009] provide a means to use

[1] Catamorphisms are a generalization of folds to tree-like data structures. We consider catamorphisms from the perspective of algebraic data types in functional programming instead of the equivalent notion in terms of functors in category theory. A catamorphism $cata_\tau\ (f_1, ..., f_n)$ replaces each occurrence of a constructor c_i of τ in a data structure with f_i. The product $(f_1, ..., f_n)$ is called an *algebra*. An element f_i of the algebra is called a *semantic function*.

types to encode properties with the expressiveness of (higher-order) intuitionistic propositional logic, and *terms* to encode proofs. Such programs are called correct by construction, because the program itself is a proof of its invariants. The goal of this chapter is therefore to apply dependent types to AGs, in order to formally reason with AGs.

Vice versa, AGs also offer benefits to dependently typed programming. Because of the Curry-Howard correspondence, dependently typed AGs are a domain-specific language to write structurally inductive proofs in a *composable*, *aspect-oriented* fashion; each attribute represents a separate aspect of the proof. Additionally, AGs alleviate the programmer from the tedious orchestration of multi-pass traversals over data structures, and ensure that the traversals are *total*: totality is required for dependently typed programs for reasons of logical consistency and termination of type checking. Hence, the combination of dependent types and AGs is mutually beneficial.

We make the following contributions in this chapter:

- We present the language AG_{DA} (Section 9.3), a light-weight approach to facilitate dependent types in AGs, and vice versa, AGs in the dependently typed language Agda. AG_{DA} is an embedding in Agda via a preprocessor.

  In contrast to conventional AGs, we can encode invariants in terms of dependently typed attributes, and proofs as values for attributes. This expressiveness comes at a price: to be able to compile to a total Agda program, we restrict ourselves to the class of ordered AGs, and demand the definitions of attributes to be total.

- We define a desugared version of AG_{DA} programs (Section 9.4) and show how to translate them to plain Agda programs (Section 9.5).

- Our approach supports a conditional attribution of nonterminals, so that we can give total definitions of what would otherwise be partially defined attributes (Section 9.6).

In Section 9.2, we introduce the notation used in this chapter. However, we assume that the reader is both familiar with AGs (see [Löh et al., 1998]) and dependently typed programming in Agda (see [Norell, 2009]).

9.2 Preliminaries

In this warm-up section, we briefly touch upon the Agda and AG notation used throughout this chapter. As an example, we implement the sum of a list of numbers with a catamorphism. We give two implementations: first one that uses plain Agda, then another using AG_{DA}. This example does not yet use dependently typed attributes. These are introduced in the next section.

In the following code snippet, the data type *List* represents a cons-list of natural numbers. The type $T'List$ is the type of the value we compute (a number), and $A'List$ is the type of an algebra for *List*. Such an algebra contains a *semantic function* for each constructor of *List*, which transforms a value of that constructor into the desired value (of type $T'List$), assuming that the transformation has been recursively applied to the fields of the constructor. The catamorphism $cata_{List}$ performs the recursive application.

```
data List : Set where            -- represents a cons-list of natural numbers
  nil  : List                    -- constructor has no fields
  cons : ℕ → List → List         -- constructor has a number and tail list as fields

T′List = ℕ                                  -- defines a type alias T′List : Set,
A′List = (T′List, ℕ → T′List → T′List)      -- and A′List : Set

cata_List : A′List → List → T′List          -- applies algebra to list
cata_List (n, _) nil  = n                   -- in case of nil, replaces nil with n
cata_List alg l  with alg | l               -- otherwise, matches on alg and l
cata_List alg l  | (_, c) | cons x xs  with cata_List alg xs  -- recurses on xs
cata_List alg l  | (_, c) | cons x xs  | r  = c x r           -- replaces cons with c
```

In Agda, a function is defined by one or more equations. A with-construct facilitates pattern matching against intermediate values. An equation that ends with with $e_1 \mid ... \mid e_n$ parameterizes the equations that follow with the values of $e_1, ..., e_n$ as additional arguments. Vertical bars separate the patterns intended for the additional parameters.

The actual algebra itself simply takes 0 for the *nil* constructor, and $\_+\_$ for the *cons* constructor. The function sum_{List} shows how the algebra and catamorphism can be used.

```
sem_nil  : T′List                        -- semantic function for nil constructor
sem_nil  = 0                             -- T′List = ℕ (defined above)

sem_cons : ℕ → T′List → T′List           -- semantic function for cons constructor
sem_cons = _+_                           -- _+_ : ℕ → ℕ → ℕ (defined in library)

sum_List : List → T′List                 -- transforms the List into the desired sum
sum_List = cata_List (sem_nil, sem_cons) -- algebra is semantic functions in a tuple
```

In the example, the sum is defined in a bottom-up fashion. By taking a function type for $T'List$, values can also be passed top-down. Multiple types can be combined by using products. Such algebras quickly become tedious to write. Fortunately, we can use AGs as a domain-specific language for algebras. In the code below, we give an AG implementation: we specify a grammar that describes the structure of the AST, declare attributes on productions, and give rules that define attributes.

We now give an implementation of the same example using AG$_{\text{DA}}$. The code consists of blocks of plain Agda code, and blocks of AG code. To ease the distinction, Agda's keywords are underlined, and keywords of AG$_{\text{DA}}$ are typeset in bold.

A grammar specification is a restricted form of a data declaration (for an AST): data constructors are called *productions* and their fields are explicitly marked as *terminal* or *nonterminal*. A nonterminal field represents a *child* in the AST and has attributes, whereas a terminal field only has a value. A plain Agda data-type declaration can be derived from a grammar specification. In such a specification, nonterminal types must have a fully saturated, outermost type constructor that is explicitly introduced by a grammar declaration. Terminal types may be arbitrary Agda types[2].

[2] In general, although not needed in this example, nonterminal types may be parametrized, production types may refer to its field names, and field types may refer to preceding field names.

```
grammar List : Set      -- declares nonterminal List of type Set
  prod nil    : List    -- production nil of type List (no fields)
  prod cons : List      -- production cons of type List (two fields)
    term        hd : ℕ      -- terminal field hd of type ℕ
    nonterm  tl   : List  -- nonterminal field tl of type List
```

With an interface specification, we declare attributes for nonterminals. Attributes come in two fashions: *inherited* attributes (used in a later example) must be defined by rules of the parent, and *synthesized* attributes may be used by the parent. Names of inherited attributes are distinct from names of synthesized attributes; an attribute of the same name and fashion may only be declared once per nonterminal. We also partition the attributes in one or more *visits*. These visits impose a partial order on attributes. Inherited attributes may not be defined in terms of a synthesized attributes of the same visit or later. We use this order in Section 9.4 to derive semantic functions that are total.

```
itf List                  -- interface for nonterminal List,
  visit compute      -- with a single visit that is named compute,
    syn sum : ℕ      -- and a synthesized attribute named sum of type ℕ
```

Finally, we define each of the production's attributes. We may refer to an attribute using *child.attr* notation. For each production, we give rules that define the inherited attributes of the children and synthesized attributes of the production itself (with *lhs* as special name), using inherited attributes of the production and synthesized attributes of the children. The special name *loc* refers to the terminals, and to local attributes that we may associate with a production.

```
datasem List     -- defines attributes of List for constructors of List
  prod nil     lhs.sum = 0                     -- rule for sum of production nil
  prod cons  lhs.sum = loc.hd + tl.sum   -- refers to terminal hd and attr tl.sum
```

The left-hand side of a rule is a plain Agda pattern, and the right-hand side is either a plain Agda expression or with-construct (not shown in this example). Additionally, both the left and right-hand sides may contain attribute references.

During attribute evaluation, visits are performed on children to obtain their associated synthesized attributes. We do not have to explicitly specify when to visit these children, neither is the order of appearance of rules relevant. However, an inherited attribute $c.x$ may not depend on a synthesized attribute $c.y$ of the same visit or later (in the interface). This guarantees that the attribute dependencies are acyclic, so that we can derive when children need to be visited and in what order.

AGs are a domain-specific language to write algebras in terms of attributes. From the grammar, we generate the data type and catamorphism. From the interface, we generate the $T'List$ type. From the rules, we generate the semantic functions sem_{nil} and sem_{cons}. AGs pay off when an algebra has many inherited and synthesized attributes. Also, there are many AG extensions that offer abstractions over common usage patterns (not covered in this chapter). In the next section we present AGs with dependent types, so that we can formulate properties of attributes (and their proofs).

9.3 Dependently Typed Example

In this section, we use $\mathrm{AG}_{\mathrm{DA}}$ to implement a mini-compiler that performs name checking of a simple language *Source*, and translates it to target language *Target* if all used identifiers are declared, or produces errors otherwise. A term in *Source* is a sequence of identifier definitions and identifier uses, for example: $\mathit{def}\ a \diamond \mathit{use}\ b \diamond \mathit{use}\ a$. In this case, b is not defined, thus the mini-compiler reports an error. Otherwise, it generates a *Target* term, which is a clone of the *Source* term that additionally carries evidence that the term is free of naming errors. Section 9.3.2 shows the definition of both *Source* and *Target*.

We show how to prove that the mini-compiler produces only correctly named *Target* terms and errors messages that only mention undeclared identifiers. The proofs are part of the implementation's code. Name checking is only a minor task in a compiler. However, the example shows many aspects of a more realistic compiler.

9.3.1 Support Code Dealing With Environments

We need some Agda support code to deal with environments. We show the relevant data structures and type signatures for operations on them, but omit the actual implementation. See Section 9.A for more details about the actual implementation. We represent the environment as a cons-list of identifiers.

$$\begin{array}{lll} \mathit{Ident} & = \mathit{String} & \text{-- } \mathit{Ident} : \mathit{Set} \\ \mathit{Env} & = \mathit{List}\ \mathit{Ident} & \text{-- } \mathit{Env} : \mathit{Set} \end{array}$$

In intuitionistic type theory, a data type represents a relation, its data constructors deduction rules for such a relation, and values built using these constructors are proofs for instances of the relation. We use some data types to reason with environments.

A value of type $\iota \in \Gamma$ is a proof that an identifier ι is member of an environment Γ. A value *here* indicates that identifier is at the front of the environment. A value *next* means that the identifier can be found in the tail of the environment, as described by the remainder of the proof.

$$\begin{array}{l} \underline{\mathit{data}}\ \_ \in \_ : \mathit{Ident} \to \mathit{Env} \to \mathit{Set}\ \underline{\mathit{where}} \\ \quad \mathit{here} : \{\iota : \mathit{Ident}\}\ \{\Gamma : \mathit{Env}\} \to \iota \in (\iota :: \Gamma) \\ \quad \mathit{next} : \{\iota_1 : \mathit{Ident}\}\ \{\iota_2 : \mathit{Ident}\}\ \{\Gamma : \mathit{Env}\} \to \iota_1 \in \Gamma \to \iota_1 \in (\iota_2 :: \Gamma) \end{array}$$

The type $\Gamma_1 \sqsubseteq \Gamma_2$ represents a proof that an environment Γ_1 is contained as a substring (with each mapping as a symbol) of an environment Γ_2. A value *subLeft* means that the environment Γ_1 is a prefix of Γ_2, and *subRight* means that Γ_1 is a suffix. With *trans*, we transitively compose two proofs.

$$\begin{array}{lll} \multicolumn{3}{l}{\underline{\mathit{data}}\ \_ \sqsubseteq \_ : \mathit{Env} \to \mathit{Env} \to \mathit{Set}\ \underline{\mathit{where}}} \\ \quad \mathit{subLeft} & : & \{\Gamma_1 : \mathit{Env}\}\ \{\Gamma_2 : \mathit{Env}\} \to \Gamma_1 \sqsubseteq (\Gamma_1 +\!\!+ \Gamma_2) \\ \quad \mathit{subRight} & : & \{\Gamma_1 : \mathit{Env}\}\ \{\Gamma_2 : \mathit{Env}\} \to \Gamma_2 \sqsubseteq (\Gamma_1 +\!\!+ \Gamma_2) \\ \quad \mathit{trans} & : & \{\Gamma_1 : \mathit{Env}\}\ \{\Gamma_2 : \mathit{Env}\}\ \{\Gamma_3 : \mathit{Env}\} \to \Gamma_1 \sqsubseteq \Gamma_2 \to \Gamma_2 \sqsubseteq \Gamma_3 \to \Gamma_1 \sqsubseteq \Gamma_3 \end{array}$$

The following functions operate on proofs. When an identifier occurs in an environment, function *inSubset* produces a proof that the identifier is also in the superset of the environment. Given an identifier and an environment, $\iota \in_? \Gamma$ returns either a proof $\iota \in \Gamma$ that the element is in the environment, or a proof that it is not.

$$\begin{array}{ll} \mathit{inSubset} & : \{\iota : \mathit{Ident}\}\ \{\Gamma_1 : \mathit{Env}\}\ \{\Gamma_2 : \mathit{Env}\} \to \Gamma_1 \sqsubseteq \Gamma_2 \to \iota \in \Gamma_1 \to \iota \in \Gamma_2 \\ \_ \in_? \_ & : (\iota : \mathit{Ident}) \to (\Gamma : \mathit{Env}) \to \neg(\iota \in \Gamma) \uplus (\iota \in \Gamma) \end{array}$$

A value of the sum-type $\alpha \uplus \beta$ either consists of an α wrapped in a constructor inj_1 or of a β wrapped in inj_2.

9.3.2 Grammar of the Source and Target Language

Below, we give a grammar for both the *Source* and *Target* language, such that we can analyze their ASTs with AGs[3]. The *Target* language is a clone of the *Source* language, except that terms that have identifiers carry a field *proof* that is evidence that the identifiers are properly introduced.

```
grammar Root      : Set          -- start symbol of grammar and root of AST
  prod root : Root    nonterm top : Source   -- top of the Source tree

grammar Source    : Set          -- grammar for nonterminal Source
  prod use        : Source       -- 'result type' of production
    term     ι    : Ident        -- terminals may have arbitrary Agda types
  prod def        : Source       -- 'result type' may be parametrized
    term     ι    : Ident
  prod _◇_        : Source       -- represents sequencing of two Source terms
    nonterm left  : Source       -- nonterminal fields must have a nonterm as
    nonterm right : Source       -- outermost type constructor.

grammar Target    : Env → Set    -- grammar for nonterminal Target
  prod def        : Target Γ     -- production type may refer to any field,
    term?    Γ    : Env          -- e.g. Γ. Agda feature: implicit terminal
    term     ι    : Ident        --   (inferred when building a def)
    term     φ    : ι ∈ Γ        -- field type may refer to preceding fields
  prod use        : Target Γ
    term?    Γ    : Env          -- a Target term carries evidence: a
    term     ι    : Ident        -- proof that the identifier is in the
    term     φ    : ι ∈ Γ        -- environment
  _◇_             : Target Γ
    term?    Γ    : Env
    nonterm left  : Target Γ     -- nonterm fields introduce children that
    nonterm right : Target Γ     -- have attributes

data Err : Env → Set where       -- data type for errors in Agda notation
```

[3] In our example, we could have defined the type *Target* using conventional Agda notation instead. However, the grammar for *Target* serves as an example of a parameterized nonterminal.

$$scope : \{\Gamma : Env\}\ (\iota : Ident) \to \neg(\iota \in \Gamma) \to Err\ \Gamma$$
$$Errs\ \Gamma = List\ (Err\ \Gamma) \qquad \text{-- } Errs : Env \to Set$$

As shown in Section 9.2, we generate Agda data-type definitions and catamorphisms from this specification.

The concrete syntax of the source language *Source* and target language *Target* of the mini-compiler is out of scope for this chapter; the grammar defines only the abstract syntax. Similarly, we omit a formal operational semantics for *Source* and *Target*: it evaluates to unit if there is an equally named *def* for every *use*, otherwise evaluation diverges.

9.3.3 Dependent Attributes

In this section, we define *dependently typed* attributes for *Source*. Such a type may contain references to preceding[4] attributes using *inh.attrNm* or *syn.attrNm* notation, which explicitly distinguishes between inherited and synthesized attributes. The type specifies a property of the attributes it references; an attribute with such a type represents a proof of this property.

In our mini-compiler, we compute bottom-up a synthesized attribute *gathEnv* that contains identifiers defined by the *Source* term. At the root, the *gathEnv* attribute contains all the defined identifiers. We output its value as the synthesized attribute *finEnv* (final environment) at the root. Also, we pass its value top-down as the inherited attribute *finEnv*, such that we can refer to this environment deeper down the AST. We also pass down an attribute *gathInFin* that represents a proof that the final environment is a superset of the gathered environment. When we know that an identifier is in the gathered environment, we can thus also find it in the final environment. We pass up the attribute *outcome*, which consists either of errors, or of a correct *Target* term.

```
itf Root          -- attributes for the root of the AST
  visit compile   syn finEnv    : Env
                  syn outcome   : (Errs syn.finEnv) ⊎ (Target syn.finEnv)
itf Source        -- attributes for Source
  visit analyze   syn gathEnv   : Env   -- attribute of first visit
  visit translate inh finEnv    : Env   -- attributes of second visit
                  inh gathInFin : syn.gathEnv ⊑ inh.finEnv
                  syn outcome   : (Errs inh.finEnv) ⊎ (Target inh.finEnv)
itf Target Γ      -- interface for Target (parameterized) is not used in the example.
```

As we show later, at the root, we need the value of *gathEnv* to define *finEnv*. This requires *gathEnv* to be placed in a strict earlier visit. Hence we define two visits, ordered by appearance.

Attribute *gathInFin* has a dependent type: it specifies that *gathEnv* is a substring of *finEnv*. A value of this attribute is a proof that essentially states that we did not forget any identifiers.

[4] We may refer to an attribute that is declared earlier (in order of appearance) in the same interface. There is one exception due to the translation to Agda (Section 9.5): in the type of an inherited attribute, we may not refer to synthesized attributes of the same visit.

Similarly, in order to construct *Target* terms, we need to prove that *finEnv* defines the identifiers that occur in the term. In the next section, we construct such proofs by applying data constructors. We may use inherited attributes as *assumptions* and pattern matches against values of attributes as *case distinctions*. Thus, with a dependently typed AG we can formalize and prove correctness properties of our implementation. Agda's type checker validates such proofs using symbolic evaluation driven by unification.

9.3.4 Semantics of Attributes

For each production, we give definitions for the declared attributes via rules. At the root, we pass the gathered environment back down as final environment. Thus, these two attributes are equal, and we can trivially prove that the final environment is a substring using either *subRight* or *subLeft*.

```
datasem Root prod root               -- rules for production root of nonterm Root
  top.finEnv      = top.gathEnv      -- pass gathered environment down
  top.gathInFin   = subRight {[]}    -- substring proof, using: [] ++ Γ4 ≡ Γ4
  lhs.finEnv      = top.gathEnv      -- pass gathEnv up
  lhs.outcome     = top.outcome      -- pass outcome up
```

For the *use*-production of *Source*, we check if the identifier (terminal *loc.ι*) is in the environment. If it is, we produce a *Target* term as value for the outcome attribute, otherwise we produce a *scope* error. For *def*, we introduce an identifier in the gathered environment. No errors can arise, hence we always produce a *Target* term. We prove ($loc.\phi_1$) that the identifier *loc.ι* is actually in the gathered environment, and prove ($loc.\phi_2$) using *inSubset* and attribute *lhs.gathInFin* that it must also be in the final environment. For _◇_, we pass *finEnv* down to both children, concatenate their *gathEnv*s, and combine their *outcome*s.

```
datasem Source                              -- rules for productions of Source
  prod use
    lhs.gathEnv     = []                                  -- no names introduced
    lhs.outcome     with loc.ι ∈? lhs.finEnv              -- tests presence of ι
                    | inj₁ notIn = inj₁ [scope loc.ι notIn]   -- when not in env
                    | inj₂ isIn  = inj₂ (use loc.ι isIn)      -- when in env
  prod def
    lhs.gathEnv     = [loc.ι]                             -- one name introduced
    loc.φ₁          = here {loc.ι} {syn.lhs.gathEnv}      -- proof of ι in gathEnv
    loc.φ₂          = inSubset lhs.gathInFin loc.φ₁       -- proof of ι in finEnv
    lhs.outcome     = inj₂ (def loc.ι loc.φ₂)             -- never any errors
  prod _◇_
    lhs.gathEnv     = left.gathEnv ++ right.gathEnv       -- pass names up
    left.finEnv     = lhs.finEnv                          -- pass finEnv down
    right.finEnv    = lhs.finEnv                          -- pass finEnv down
    left.gathInFin  = trans subLeft lhs.gathInFin         -- proof for left
```

```
right.gathInFin = trans (subRight {syn.lhs.gathEnv} {lhs.finEnv})
                             lhs.gathInFin                    -- proof for right

lhs.outcome    with left.outcome                              -- four alts.
               | inj₁ es  with right.outcome
               | inj₁ es₁ | inj₁ es₂ = inj₁ (es₁ ++ es₂)     -- 1: both in error
               | inj₁ es₁ | inj₂ _   = inj₁ es₁              -- 2: only left
               | inj₂ t₁  with left.outcome
               | inj₂ t₁  | inj₁ es₂ = inj₁ es₂              -- 3: only right
               | inj₂ t₁  | inj₂ t₂  = inj₂ (t₁ ◇ t₂)         -- 4: none in error
```

Out of the above code, we generate each production's semantic function (and some wrapper code), such that these together with a catamorphism form a function that translates *Source* terms. The advantage of using AGs here is that we can easily add more attributes (and thus more properties and proofs) and refer to them.

9.4 AG Descriptions and their Core Representation

In the previous sections, we presented AG_{DA} (by example). To describe the dependently-typed extension to AGs, we do so in terms of the core language $\text{AG}^{\text{X}}_{\text{DA}}$ (a subset of AG_{DA}). Implicit information in AG descriptions (notational conveniences, the order of rules, visits to children) is made explicit in $\text{AG}^{\text{X}}_{\text{DA}}$. We sketch the translation from AG_{DA} to $\text{AG}^{\text{X}}_{\text{DA}}$. In previous work [Middelkoop et al., 2010c,a], we described the process in more detail (albeit in a non-dependently typed setting).

$\text{AG}^{\text{X}}_{\text{DA}}$ contains interface declarations, but grammar declarations are absent and semantic blocks encoded differently. Each production in AG_{DA} is mapped to a *semantic function* in $\text{AG}^{\text{X}}_{\text{DA}}$: it is a domain-specific language for the contents of semantic functions. A terminal $x : \tau$ of the production is mapped to a parameter $loc_t x : \tau$. Implicit terminals are mapped to implicit parameters. A nonterminal $x : N\ \overline{\tau}$ is mapped to a parameter $loc_c x : T'N\ \overline{\tau}$. The body of the production consists of the rules for the production given in the original $\text{AG}^{\text{X}}_{\text{DA}}$ description, plus a number of additional rules that declare children and their visits explicitly.

```
sem◇ : T′Source → T′Source → T′Source    -- derived from (non)terminal types
sem◇ loc_c left loc_c right =             -- semantic function for ◇
  sem : Source                          -- AG^X_DA semantics block
    child left : Source  = loc_c left    -- defines a child left
    child right : Source = loc_c right   -- defines a child right
    invoke analyze   of left            -- rule requires visiting analyze on left
    invoke analyze   of right           -- rule requires visiting analyze on right
    invoke translate of left
    invoke translate of right
    lhs.gathEnv = left.gathEnv ++ right.gathEnv   -- the AG_DA rules
    ...                                           -- etc.
```

$e ::= \textsc{Agda}\ [\overline{b}]$	-- embedded blocks b in AGDA
$b ::= i \mid s \mid o$	-- $\text{AG}^{\text{X}}_{\text{DA}}$ blocks
$o ::= inh.c.x \mid syn.c.x \mid loc.x$	-- embedded attribute reference
$i ::= \textbf{itf}\ I\ \overline{x} : \tau\ v$	-- with first visit v, params x, and signature τ
$v ::= \textbf{visit}\ x\ \textbf{inh}\ \overline{a}\ \textbf{syn}\ \overline{a}\ v$	-- visit declaration
$\mid\ \square$	-- terminator of visit decl. chain
$a ::= x : e$	-- attribute decl, with Agda type e
$s ::= \textbf{sem} : I\ \overline{e}\ t$	-- semantics expr, uses interface $I\ \overline{e}$
$t ::= \textbf{visit}\ x\ \overline{r}\ t$	-- visit definition, with next visit t
$\mid\ \square$	-- terminator of visit def. chain
$r ::= p\ e'$	-- evaluation rule
$\mid\ \textbf{invoke}\ x\ \textbf{of}\ c$	-- invoke-rule, invokes x on child c
$\mid\ \textbf{child}\ c : I = e$	-- child-rule, defines a child c, with interface $I\ \overline{e}$
$p ::= o$	-- attribute def
$\mid\ .\{e\}$	-- Agda dot pattern
$\mid\ x\ \overline{p}$	-- constructor match
$e' ::= \underline{with}\ e\ \overline{p'\ e'^{?}}$	-- Agda $\underline{with}$ expression (e' absent when p' absurd)
$\mid\ = e$	-- Agda $=$ expression
p'	-- Agda LHS
x, I, c	-- identifiers, interface names, children respectively
τ	-- plain Agda type

Figure 9.1: Syntax of RULER-CORE

A child rule introduces a child with explicit semantics (a value of the type $T'Source$). Other rules may declare visits and refer to the attributes of the child. An invoke rule declares a visit to a child, and brings the attributes of that visit in scope. Conventional rules define attributes, and may refer to attributes. The dependencies between attributes induces a def-use (partial) order.

Actually, there is one more step to go to end up with a $\text{AG}^{\text{X}}_{\text{DA}}$ description. A semantics block consists of one of more visit-blocks (in the order specified by the interface), and the rules are partitioned over the blocks. In a block, the *lhs* attributes of that and earlier visits are in scope, as well as those brought in scope by preceding rules. Also, the synthesized attributes of the visit must be defined in the block or in an earlier block. We assign rules to the earliest block that satisfies the def-use order. We convert this partial order into a total order by giving conventional rules precedence over child/invoke rules, and using the order of appearance otherwise:

$sem\diamond : T'Source \rightarrow T'Source \rightarrow T'Source$	-- signature derived from itf
$sem\diamond\ loc_c left\ loc_c right =$	-- semantic function for $\diamond$

```
sem : Source                                  -- AG$^X_{DA}$ block
  visit analyze                               -- first visit
    child left : Source   = loc_c left        -- defines a child left
    invoke analyze  of left                   -- requires child to be defined
    child right : Source  = loc_c right       -- defines a child right
    invoke analyze  of right                  -- requires child to be defined
    syn.lhs.gathEnv = syn.left.gathEnv ++ syn.right.gathEnv
  visit translate                             -- second visit
    inh.left.finEnv      = inh.lhs.finEnv     -- needs lhs.finEnv
    inh.right.finEnv     = inh.lhs.finEnv     -- needs lhs.finEnv
    inh.left.gathInFin   = trans ...          -- also needs lhs.gathEnv
    inh.right.gathInFin  = trans ...          -- also needslhs.gathEnv
    invoke translate of left                  -- needs def of inh attrs of left
    invoke translate of right                 -- needs def of inh attrs of right
    syn.lhs.outcome  with ...                 -- needs translate attrs of children
```

It is a static error when such an order cannot be satisfied. Another interesting example is the semantic function for the root: it has a child with an interface different from its own, and has two invoke rules in the same visit.

```
sem_root : T′Source → T′Root                  -- semantic function for the root
sem_root locStop =                            -- Source's semantics as parameter
  sem : Root visit compile                    -- only one visit
    child top : Source = loc_c top            -- defines a child top
    invoke analyze of top                     -- invokes first visit of top
    inh.top.finEnv    = syn.top.gathEnv       -- passes gathered environment back
    invoke translate of top                   -- invokes second visit of top
    syn.lhs.output    = syn.top.gathEnv       -- passes up the gathered env
    syn.lhs.output    = syn.top.outcome       -- passes up the result
```

Figure 9.1 shows the syntax of AG$^X_{DA}$. In general, interfaces may be parametrized. The interface has a function type τ (equal to the type of the nonterminal declaration in AG$_{DA}$) that specifies the type of each parameter, and the kind of the interface (an upper bound of the kinds of the parameters). For an evaluation rule, we either use a with-expression when the value of the attribute is conditionally defined, or use a simple equation as RHS. In the next section, we plug such an expression in a function defined via with-expressions; hence we need knowledge about the with-structure of the RHS.

9.5 Translation to Agda

To explain the preprocessing of AG$^X_{DA}$ to Agda, we give a translation scheme in Figure 9.2 (explained via examples below). This translation scheme is a denotational semantics for AG$^X_{DA}$. Also, if the translation is correct Agda, then the original is correct AG$^X_{DA}$.

$$
\begin{array}{lll}
\llbracket \mathbf{itf}\ I\ \overline{x} : \overline{\tau_x} \to \tau \rrbracket\ v & \leadsto \llbracket_{\mathsf{iv}}\ v \rrbracket_{I,\tau}^{\overline{x:\tau_x}}\ \ ;\ \ \llbracket sig\ I \rrbracket : \llbracket \tau \rrbracket\ \ ;\ \ \llbracket sig\ I \rrbracket = \llbracket sig\ I\ (name\ v) \rrbracket & \\
\llbracket_{\mathsf{iv}}\ \mathbf{visit}\ x\ \mathbf{inh}\ \overline{a}\ \mathbf{syn}\ \overline{b}\ v \rrbracket_{I,\tau}^{\overline{g}} & \leadsto \llbracket_{\mathsf{iv}}\ v \rrbracket_{I,\tau}^{\overline{g} +\!\!+ \overline{a} +\!\!+ \overline{b}} & \text{-- interface type for later visits} \\
 & \quad \llbracket sig\ I\ x \rrbracket : \llbracket_{\mathsf{at}}\ g_1 \rrbracket \to \ldots \to \llbracket_{\mathsf{at}}\ g_n \rrbracket \to \llbracket resultty\ \tau \rrbracket & \\
 & \quad \llbracket sig\ I\ x \rrbracket\ \overline{\llbracket_{\mathsf{an}}\ g \rrbracket} = \llbracket_{\mathsf{a}}\ inh.a_1 \rrbracket \to \ldots \to \llbracket_{\mathsf{a}}\ inh.a_n \rrbracket \to & \\
 & \qquad\qquad \llbracket typrod\ (\overline{syn}.b)\ (sig\ I\ (name\ v)) \rrbracket & \\
\llbracket_{\mathsf{iv}}\ \square \rrbracket_{I,\tau}^{\overline{g}} & \leadsto \llbracket sig\ I\ \square \rrbracket = \square & \text{-- terminator (some unit-value)} \\
\llbracket_{\mathsf{a}}\ x : e \rrbracket & \leadsto \llbracket atname\ x \rrbracket : \llbracket e \rrbracket & \text{-- extract attribute name and type} \\
\llbracket_{\mathsf{at}}\ x : e \rrbracket & \leadsto \llbracket e \rrbracket & \text{-- extract attribute type} \\
\llbracket_{\mathsf{an}}\ x : e \rrbracket & \leadsto \llbracket atname\ x \rrbracket & \text{-- extract attribute name} \\
\llbracket \mathbf{sem}\ x : I\ \overline{e}\ t \rrbracket & \leadsto \llbracket vis\ lhs\ (name\ t) \rrbracket\ \underline{where}\ \llbracket_{\mathsf{ev}} t \rrbracket_{I}^{\overline{e},\emptyset} & \text{-- top of semfun} \\
\llbracket_{\mathsf{ev}} \mathbf{visit}\ x\ \overline{r}\ t \rrbracket_{I}^{\overline{e},\overline{g}} & \leadsto \llbracket vis\ lhs\ x \rrbracket : \llbracket sig\ I\ x \rrbracket\ \llbracket \overline{e} \rrbracket\ \overline{\llbracket_{\mathsf{an}}\ g \rrbracket} & \text{-- type of visit fun} \\
 & \quad \llbracket vis\ lhs\ x \rrbracket\ \llbracket inhs\ I\ x \rrbracket = \llbracket_{\mathsf{r}}\ \overline{r} \rrbracket_{\llbracket \varsigma \rrbracket} & \text{-- chain of rules} \\
 & \quad \llbracket \varsigma \rrbracket \leadsto\ = \llbracket valprod\ (syns\ I\ x)\ (vis\ lhs\ (name\ t)) \rrbracket & \\
 & \qquad\quad \underline{where}\ \llbracket_{\mathsf{ev}} t \rrbracket_{I}^{\overline{g} +\!\!+ \overline{a} +\!\!+ \overline{b}} & \text{-- next visit} \\
\llbracket_{\mathsf{ev}} \square \rrbracket_{I}^{\overline{e},\overline{g}} & \leadsto \llbracket vis\ lhs\ \square \rrbracket : \llbracket sig\ I\ \square \rrbracket\ \llbracket \overline{e} \rrbracket\ \overline{\llbracket_{\mathsf{an}}\ g \rrbracket}\ \ ;\ \ \llbracket vis\ lhs\ \square \rrbracket = \square & \\
\llbracket_{\mathsf{r}}\ \mathbf{child}\ c : I = e \rrbracket_k & \leadsto \underline{with}\ \llbracket e \rrbracket\ \ \ldots \mid \llbracket vis\ I\ (firstvisit\ I) \rrbracket\ \llbracket k \rrbracket & \text{-- } k\text{: remaining rules} \\
\llbracket_{\mathsf{r}}\ \mathbf{invoke}\ x\ \mathbf{of}\ c \rrbracket_k & \leadsto \underline{with}\ \llbracket vis\ (itf\ c)\ x \rrbracket\ \llbracket inhs\ (itf\ c)\ x \rrbracket & \text{-- pass inh values} \\
 & \quad \ldots \mid (valprod\ (syns\ (itf\ c)\ x))\ \llbracket k \rrbracket & \text{-- match syn values} \\
\llbracket_{\mathsf{r}}\ p\ e' \rrbracket_k & \leadsto \llbracket_{\mathsf{ep}}\ e' \rrbracket_{p}^{k} & \text{-- translation for attr def rule} \\
\llbracket_{\mathsf{ep}}\ \underline{with}\ e\ \overline{p\ e'} \rrbracket_{p}^{k} \leadsto \underline{with}\ e & \ \ \overline{\ldots \mid \llbracket p \rrbracket\ \llbracket_{\mathsf{r}}\ p\ e' \rrbracket_k} & \text{-- rule RHS is with-constr} \\
\llbracket_{\mathsf{ep}} = e \rrbracket_{p}^{k} & \leadsto \underline{with}\ e\ \ \ldots \mid \llbracket p \rrbracket\ k & \text{-- rule RHS is expr}
\end{array}
$$

$$
\begin{array}{lll}
atref\ inh.c.x = c_i x & atname\ inh.x = inh_a x & \text{-- naming conventions} \\
atref\ syn.c.x = c_s x & atname\ syn.x = syn_a x & \text{-- } atref\text{: ref to attr value} \\
atref\ loc.x\ \ = loc_l x & atname\ x\ \ \ \ \ = x & \text{-- } atname\text{: ref to attr in type} \\
vis\ I\ x = vis\ lhs\ x & sig\ I\ \ = T' I & \text{-- } vis\text{: name of visit function} \\
vis\ c\ x = c_v x & sig\ I\ x = T' I' x & \text{-- } sig\text{: itf types}
\end{array}
$$

Figure 9.2: Translation of $\mathrm{AG}_{\mathrm{DA}}^{\mathrm{X}}$ to Agda.

A semantics block in an $\mathrm{AG}^{\mathrm{X}}_{\mathrm{DA}}$ program is actually an algorithm that makes precise how to compute the attributes as specified by the interface: for each visit, the rules prescribe when to compute an attribute and when to visit a child. The idea is that we map such a block to an Agda function that takes values for its inherited attributes and delivers a dependent product[5] of synthesized attributes. However, such a function would be cyclic: in the presented example, the result *gathEnv* would be needed for as input for *finEnv*. Fortunately, we can bypass this problem: we map to a *k*-visit *coroutine* instead. A coroutine is a function that can be invoked *k* times. We associate each invocation with a visit of the interface. Values for the inherited attributes are inputs to the invocation. Values for the synthesized attributes are the result of the invocation. In a pure functional language (like Agda), we can encode coroutines as one-shot continuations (or *visit functions* [Saraiva and Swierstra, 1999]).

We generate types for coroutines and for the individual visit functions that make up such a coroutine. These types are derived from the interface. For each visit (e.g. *translate* of *Source*), we generate a type that represents a function type from the attribute types of the inherited attributes for that visit, to a dependent product (Σ) of the types of the synthesized attributes and the type of the next visit function. These types are parameterized with the attributes of earlier visits (e.g. $T'Source'translate\ syn_a gathEnv$). The type of the coroutine itself is the type of the first visit. The type of the last visit is a terminator $\Box$.

$$
\begin{aligned}
&T'Source &&= T'Source'analyze\\
&T'Source'analyze &&= \Sigma\ \ Env\ \ T'Source'translate
\end{aligned}
$$

$$
\begin{aligned}
&T'Source'translate\ \ syn_a gathEnv\ =\\
&\quad (inh_a finEnv : Env)\ \rightarrow\ (inh_a gathInFin : syn_a gathEnv \sqsubseteq inh_a finEnv)\ \rightarrow\\
&\qquad \Sigma\ (Errs\ inh_a finEnv \uplus Target\ inh_a finEnv)\\
&\qquad\quad (T'Source'\Box\ \ syn_a gathEnv\ inh_a finEnv\ inh_a gathInFin)\\
&T'Source'\Box\ \ syn_a gathEnv\ inh_a finEnv\ inh_a gathInFin\ syn_a outcome\ = \Box
\end{aligned}
$$

The restrictions on attribute order in the interface ensure that referenced attributes are in scope. This representation can be optimized a bit by passing only on those attributes that are referenced in the remainder. The scheme for $[\![_{\mathsf{iv}}\ v]\!]^{I}_{g,\tau}$ formalizes this translation, where g is the list of preceding attribute declarations, and τ the type for I. The *typrod* function mentioned in the scheme constructs a right-nested dependent product.

The coroutine itself consists of nested continuation functions (one for each visit). Each continuation takes the visit's inherited attributes as parameter, and consists of a tree of with-constructs that represent intermediate computations for computations of attributes and invocations of visits to children. Each leaf ends in a dependent product of the visit's synthesized attributes and the continuation function for the next visit[6].

```
sem◇ : T'Source → T'Source → T'Source                 -- example translation for ◇
sem◇ loc_c left loc_c right = lhs_v analyze where       -- delegates to first visit function
   lhs_v analyze : T'Source'analyze                    -- signature of first visit function
   lhs_v analyze with ...                              -- computations for analyze here
```

[5] A dependent product $\Sigma\ \tau\ f = (\tau, f\ \tau)$ parameterizes the RHS f with the LHS τ.

[6] As a technical detail, a leaf of the with-tree may also be an *absurd pattern*. These are used in Agda to indicate an alternative that is never satisfiable. A body for such an alternative cannot be given.

```
... = (lhs_s gathEnv, lhs_v translate) ahwere        -- result of first visit function
  lhs_v translate : T′Source′translate lhs_s gathEnv  -- last visit function
  lhs_v translate lhs_i finEnv lhs_i gathInFin with ... -- computations for translate here
  ... = (lhs_s outcome, lhs_v □) where                -- result of second visit function
    lhs_v □ : T′Source′□ lhs_s gathEnv lhs_i finEnv lhs_i gathInFin lhs_s outcome
    lhs_v □ = □                                        -- explicit terminator value
```

The scheme $[\![_{\mathsf{ev}} v]\!]_I^{\overline{e},\overline{g}}$ formalizes this translation for a visit v of interface I, where $\overline{e}$ are type arguments to the interface (empty in the example), and $\overline{g}$ are the attributes of previous visits.

The with-tree for a visit-function consists of the translation of child-rules, invoke-rules and evaluation rules. Each rule plugs into this tree. For example, the translation for [[**child** *left* : *Source* = *loc*$_s$*left*]] is:

```
... with loc_s left            -- evaluate RHS to get first visit fun
... | left_v analyze with ...  -- give it a name + proceed with remainder
```

For [[**invoke** *translate* **of** *left*]] the translation is:

```
... with left_v translate left_i finEnv left_i gathInFin   -- visit fun takes inh attrs
... | (left_s outcome, left_v sentinel) with ...           -- returns product of syn attrs
```

For [[*lhs.gathEnv* = *left.gathEnv* ++ *right.gathEnv*]]:

```
... with left_s gathEnv ++ right_s gathEnv   -- translation for RHS
... | lhs_s gathEnv with ...                 -- LHS + remainder
```

For [[*lhs.outcome* with...]] (where the RHS is a with-construct), we duplicate the remaining with-tree for each alternative of the RHS:

```
... with left_s outcome                              -- translation for RHS
... | inj₁ es  with right_s outcome
... | inj₁ es₁ | inj₁ es₂ with inj₁ (es₁ ++ es₂)     -- alternative one of four
... | inj₁ es₁ | inj₁ es   | lhs_s outcome with ...  -- LHS + remainder
... | inj₁ es₁ | inj₂ _   with inj₁ es₁              -- alternative two of four
... | inj₁ es₁ | inj₂ _    | lhs_s outcome with ...  -- LHS + remainder
... | inj₂ ...                                       -- remaining two alternatives
```

The scheme $[\![_{\mathsf{r}}\, r]\!]_k$ formalizes this translation, where r is a rule and k the translation of the rules that follow r.

The size of the translated code may be exponential in the number of rules with with-constructs as RHS. It is not obvious how to treat such rules otherwise. Agda does not allow a with-construct as a subexpression. Neither can we easily factor out the RHS of a rule to a separate function, because the conclusions drawn from the evaluation of preceding rules are not in scope of this function. Fortunately, for rules that would otherwise cause a lot of needless duplication, the programmer can perform this process manually.

When dependent pattern matching brings assumptions in scope that are needed *across* rules, the code duplication is a necessity. To facilitate that pattern matching effects are visible across rules, we need to ensure that the rule that performs the match is ordered before a rule that needs the assumption. Chapter 3 shows how such non-attribute dependencies can be captured.

The translated code has attractive operational properties. Each attribute is only computed once, and each node is at most traversed *k* times.

9.6 Partially Defined Attributes

A fine granularity of attributes is important to use an AG effectively. In the mini-compiler of Section 9.3, we could replace the attribute *outcome* with an attribute *code* and a separate attribute *errors*. This would be more convenient, since it would not require a pattern match against the *output* attribute to collect errors. This is convenient in general, as a finer granularity of attributes gives more opportunities to use default rules. However, we cannot produce a target term in the presence of errors, thus *code* would not have a total definition. Therefore, we were forced to combine these two aspects into a single attribute *outcome*. It is common to use partially defined attributes in an AG. This holds especially when the attribute's value (e.g. *errors*) determines if another attribute is defined (e.g. *code*). We present a solution that uses the partitioning of attributes over visits.

The idea is to make the availability of visits dependent on the value of a preceding attribute. We split up the *translate* visit in a visit *report* and a visit *generate*. The visit *report* has *errors* as synthesized attribute, and *generate* has *code*. Furthermore, we enforce that *generate* may only be invoked (by the parent in the AST) when the list of errors reported in the previous visit is empty. We accomplish this with an additional attribute *noErrors* on *generate* that gives evidence that the list of errors is empty. With this evidence, we can give a total definition for *code*.

```
itf Source     -- Root's visit needs to be split up in a similar way
  visit report     syn errors   : Errs inh.finEnv     -- parent can inspect errors
  visit generate   inh noErrors : syn.errors ≡ [ ]    -- enforces invariant
                   syn code     : Target inh.finEnv   -- only when errors is empty
datasem Source prod use     -- example for production use
  loc.testInEnv = loc.ι ∈? lhs.finEnv        -- scheduled in visit report
  lhs.code with loc.testIn | lhs.noErrors    -- scheduled in visit generate
      | inj₁ _     | ()                      -- cannot happen, hence an absurd pattern
      | inj₂ isIn  | refl = use loc.ι isIn   -- extract the evidence needed for the code term
datasem Source prod ◇    -- leftNil : (α : Env) → (β : Env) → (α ++ β ≡ [ ]) → (α ≡ [ ])
  left.noErrors = leftNil left.errors right.errors lhs.noErrors    -- right.noErrors similar
  lhs.code      = left.code ◇ right.code     -- scheduled in visit generate
```

For this approach to work, it is essential that visits are scheduled as late as possible, and only those that are needed.

Another application of the above idea is related to proofs for special cases, i.e. when we want to prove that with additional assumptions on inherited attributes, the synthesized attributes meet additional criteria. These assumptions are modeled as additional inherited attributes. However, since we are required to pass values for inherited attributes, our AG would only work for these special cases. For example, suppose that we want to prove that an AG for a type inferencer is complete. To do so, we give a typing derivation as input, and require a proof that the inferred type is more general than the type of the typing derivation. To infer a type, we do not want to provide a typing derivation, and in the proof, we do not want that the typing derivation would be defined in terms of the typing derivation. The typing derivation assumption is thus partially defined: only needed for the proof. Hence, we define the attributes for the proof in a separate visit.

```
itf Expr
  visit infer    inh env : Env   syn self : Expr   syn errs : Errs
  visit typed    inh $\phi_1$ : syn.errs $\equiv$ [ ]   syn $\tau$ : Ty
  visit proof                                    -- special visit for proof
    inh $\tau'$     : Ty
    inh deriv : inh.env $\vdash$ syn.self : inh.$\tau'$   -- description of typing derivation.
    syn $\phi_2$    : syn.$\tau \leqslant$ inh.$\tau'$
```

We can generalize the presented approach (Section 9.B) by defining a fixed number of alternative sets of attributes for a visit, and use the value of a preceding attribute to select one of these sets [Middelkoop et al., 2010d].

9.7 Related Work

Dependent types originate in Martin-Löf's Type Theory. Several dependently-typed programming languages increasingly gain popularity, including the languages Agda [Norell, 2009], Epigram [McBride, 2004], and Coq [Bertot, 2008]. We present the ideas in this chapter with Agda as host language, because it has a concept of a dependent pattern match, to which we straightforwardly map the left-hand sides of AG rules. Also, in Coq and Epigram, a program is written via interactive theorem proving with tactics or commands. The preprocessor-based approach of this chapter, however, suits a declarative approach more.

Attribute grammars [Knuth, 1968] are considered to be a promising implementation for compiler construction. Recently, many Attribute Grammar systems arose for mainstream languages, such as the systems JastAdd [Ekman and Hedin, 2007] and Silver [Wyk et al., 2008] for Java, and UUAG [Löh et al., 1998] for Haskell. These approaches may benefit from the stronger type discipline as presented in this chapter; however, it would require an encoding of dependent types in the host language.

In languages languages with meta-programming facilities, it is sometimes possible to implement AGs without the need of a preprocessor. Viera et al. [2009] show how to implement AGs into Haskell via type level programming. Each rule exposes in its type the attributes that it needs and the attributes that it defines. The rules can be composed via combinators. At one place in the program, the knit point, a proof is constructed that the set of used attributes equals the defined attributes. This proof is subsequently mapped to a semantic function. A combination of that paper with our work, would fit well in Agda, if Agda had a mechanism similar to Haskell's class system. Alternatively, it may be possible to embed first-class AGs in Agda, while using a preprocessor to generate boilerplate code.

AGs have a straightforward translation to cyclic functions in a lazy functional programming language [Swierstra and Alcocer, 1998]. To prove that cyclic functions are total and terminating is a non-trivial exercise. Kastens [Kastens, 1980] presented Ordered Attribute Grammars (OAGs). In OAGs, the evaluation order of attribute computations as well as attribute lifetime can be determined statically. Saraiva [Saraiva and Swierstra, 1999] described how to generate (noncyclic) functional coroutines from OAGs. The coroutines we generate are based on these ideas.

9.8 Conclusion

We presented AG$_{\mathrm{DA}}$, a language for ordered AGs with dependently typed attributes: the type of an attribute may refer to the value of another attribute. This feature allows us to conveniently encode invariants in the type of attributes, and pass proofs of these invariants around as attributes. With a dependently typed AG, we write algebras for catamorphisms in a dependently typed language in a composable way. Each attribute describes a separate aspect of the catamorphism.

A particular advantage of composability is that attributes can easily be added and shared. Moreover, via *local attributes* we can specify invariants and proofs at those places where the data is. We prove for the example in Section 9.3 that the final environment must be equal to the gathered environment at the root of the tree:

```
datasem Root prod root     -- more rules for the root production
  loc.eqEnvs : inh.top.finEnv ≡ syn.top.gathEnv    -- signature for local attr
  loc.eqEnvs = refl                                -- proof of the equality
```

The approach we presented is lightweight, which means that we encode AGs as an embedded language (via a preprocessor), such that type checking is deferred to the host language. To facilitate termination checking, we translate the AG to a coroutine (Section 9.5) that encodes a terminating, multi-visit traversal, under the restriction that the AG is ordered and definitions for attributes are total.

The preprocessor approach fits nicely with the interactive Emacs mode of Agda. Type errors in the generated program are traceable back to the source: in a statically checked AG$_{\mathrm{DA}}$ program these can only occur in Agda blocks. These Agda blocks are literally preserved; due to unicode, even attribute references can stay the same. Also, the Emacs mode implements interactive features via markers, which are also preserved by the translation. The AG preprocessor is merely an additional preprocessing step. Not all features integrate seamlessly, however. Syntactical errors in Agda blocks, such as an omitted closing parenthesis, may only be discovered during parsing of the generated code surrounding the block. This can be remedied by validating the syntax of Agda blocks during the preprocessing. A complication arises because the code of rules may occur multiple times in the generated code. Also, the case splitting feature causes the generated program to be transformed, such that it scrutinizes on a variable chosen by the programmer. The additional equations generated by case splitting need to be transformed back to rules with a with-construct. Fortunately, these are not fundamental problems.

With some generalizations, the work we have presented is a proposal for a more flexible termination checker for Agda that accepts *k*-orderable cyclic functions, if the function can be written as a non-cyclic *k*-visit coroutine.

As future work, it may be possible to exploit patterns in AG descriptions to generate boilerplate for proofs. For example, we can generate boilerplate code to express termination and monotonicity properties of fixpoint iteration in AGs (Chapter 5), and generate boilerplate code for standard environment lookup and extension patterns. Also, it may be possible to generate proof and AG templates from a type system specification [Dijkstra and Swierstra, 2006b].

9.A Implementation of the Support Code

In this section, we give a definition of the support code mentioned briefly in Section 9.3. This section also serves to give a bit more background information about Agda's syntax.

In a dependently typed language, the interpretation of an algebraic data type in an intuitionistic logic is a relation between the type parameters of the data type. A data constructor is an axiom for the relation. A type is a theorem; a value of that type is a proof that the theorem holds.

We defined the following data types and data constructors to work with proofs for environments. We use the $\_\in\_$ data type to prove that an identifier is in the environment, and $\_\sqsubseteq\_$ to prove that an environment occurs as substring in an environment.

$$\begin{array}{l}
\underline{\mathit{data}}\ \_\in\_ : \mathit{Ident} \to \mathit{Env} \to \mathit{Set}\ \underline{\mathit{where}} \quad \text{-- member of environment}\\
\quad \mathit{here} : \{\iota : \mathit{Ident}\}\ \{\Gamma : \mathit{Env}\} \to \iota \in (\iota :: \Gamma)\\
\quad \mathit{next} : \{\iota_1 : \mathit{Ident}\}\ \{\iota_2 : \mathit{Ident}\}\ \{\Gamma : \mathit{Env}\} \to \iota_1 \in \Gamma \to \iota_1 \in (\iota_2 :: \Gamma)\\
\underline{\mathit{data}}\ \_\sqsubseteq\_ : \mathit{Env} \to \mathit{Env} \to \mathit{Set}\ \underline{\mathit{where}} \quad \text{-- substring of environment}\\
\quad \mathit{subLeft}\ \ : \{\Gamma_1 : \mathit{Env}\}\ \{\Gamma_2 : \mathit{Env}\} \to \Gamma_1 \sqsubseteq (\Gamma_1 +\!\!+ \Gamma_2)\\
\quad \mathit{subRight} : \{\Gamma_1 : \mathit{Env}\}\ \{\Gamma_2 : \mathit{Env}\} \to \Gamma_2 \sqsubseteq (\Gamma_1 +\!\!+ \Gamma_2)\\
\quad \mathit{trans}\ \ \ \ \ \ : \{\Gamma_1 : \mathit{Env}\}\ \{\Gamma_2 : \mathit{Env}\}\ \{\Gamma_3 : \mathit{Env}\} \to \Gamma_1 \sqsubseteq \Gamma_2 \to \Gamma_2 \sqsubseteq \Gamma_3 \to \Gamma_1 \sqsubseteq \Gamma_3
\end{array}$$

A membership proof for an identifier in the environment states that either the identifier is at the head of the environment, or there is a proof that it is in the tail of the environment. The substring-proof gives prefixes and suffixes to the encapsulated environment that together give the encapsulating environment. Note the use of curly braces here. These represent implicit parameters, which are denoted with similar syntax as implicit parameters for Haskell. An argument for an implicit parameter may be omitted if it can be derived from the context via unifications.

The arrow type constructor can be interpreted as the logical implication. The parameters of a function are assumptions, and the return type is the conclusion. A function thus takes proofs for these assumption as parameter, and transforms these into a proof for the result. The following functions operate on proofs of the above types. When an identifier exists in an environment, then *append* proves that it also exists in a suffixed version of that environment. Similarly, *prefix* gives a prove for a prefixed environment. Function *inSubset* uses these two to prove that when an identifier occurs in a substring of an environment, it also occurs in the environment itself.

```
$append : \{\iota : Ident\}\ \{\Gamma : Env\} \to (\iota \in \Gamma) \to (\Gamma' : Env) \to (\iota \in (\Gamma +\!\!+ \Gamma'))$
$append\ \{\iota\}\ \{.\iota :: \Gamma\}\quad (here)\qquad \Gamma' = here\ \{\iota\}\ \{\Gamma +\!\!+ \Gamma'\}$
$append\ \{\iota\}\ \{nm' :: \Gamma\}\ (next\ in\Gamma)\ \Gamma' = next\ (append\ \{\iota\}\ \{\Gamma\}\ in\Gamma\ \Gamma')$
$append\ \{\_\}\ \{[\,]\}\qquad ()\qquad \_$
$prefix : \{\iota : Ident\}\ \{\Gamma' : Env\} \to (\iota \in \Gamma') \to (\Gamma : Env) \to (\iota \in (\Gamma +\!\!+ \Gamma'))$
$prefix\ in\Gamma'\ [\,]\qquad = in\Gamma'$
$prefix\ in\Gamma'\ (x :: \Gamma) = next\ (prefix\ in\Gamma'\ \Gamma)$
$inSubset : \{\iota : Ident\}\ \{\Gamma : Env\}\ \{\Gamma' : Env\} \to (\Gamma' \sqsubseteq \Gamma) \to \iota \in \Gamma' \to \iota \in \Gamma$
$inSubset\ (subLeft\ \{\_\}\ \{\Gamma'\})\ in\Gamma' = append\ in\Gamma'\ \Gamma'$
$inSubset\ (subRight\ \{\Gamma\})\qquad in\Gamma' = prefix\ in\Gamma'\ \Gamma$
$inSubset\ (trans\ subL\ subR)\quad in\Gamma' = inSubset\ subR\ (inSubset\ subL\ in\Gamma')$
```

In case of *append*, the environment cannot be empty when we have a proof than an identifier occurs in it. However, to satisfy the totality checker, we are required to give a function definition for this case. Since a match against such a pattern cannot succeed, the match is called absurd, and no function body has to be given.

The operator $\in_?$ takes an identifier ι and an environment Γ, and either gives a prove that the identifier is in the environment, or gives a proof that it is not in the environment. The sum type $\_\uplus\_$ (named *Either* in Haskell) provides constructors inj_1 (*Left* in Haskell) and inj_2 (*Right* in Haskell) for this purpose.

For the definition of $\in_?$, we use function *notFirst* to prove by contradiction that if an identifier does not occur in the tail of the environment, and is also not equal to the head of the environment, that it is neither in the whole environment.

```
$notFirst : \{\iota : Ident\}\ \{nm' : Ident\}\ \{\Gamma : Env\} \to$
  $\neg(\iota \equiv nm') \to \neg(nm' \in \Gamma) \to \neg(nm' \in (\iota :: \Gamma))$
$notFirst\ \phi_1\ \_\ = \lambda here\qquad \to \phi_2\ refl$
$notFirst\ \_\ \ \phi_1 = \lambda (next\ \phi_2) \to \phi_1\ \phi_2$
```

Negation of a type (neg tau) is defined as a function $\tau \to \bot$. The type $\bot$ (falsum) does not have any data constructors, so we cannot construct it explicitly, but can match against it with an absurd pattern. If we can derive its value, we proved a contradiction between the assumptions we made.

```
$\_ \in_? \_ : (\iota : Ident) \to (\Gamma : Env) \to \neg(\iota \in \Gamma) \uplus (\iota \in \Gamma)$
$nm' \in_? [\,] = inj_1 \lambda ()$
$nm' \in_? (\iota :: \Gamma)\quad \underline{with}\ \iota \equiv_? nm'$
$nm' \in_? (.nm' :: \Gamma)\ |\ yes\ refl = inj_2\ here$
$nm' \in_? (\iota :: \Gamma)\quad\ |\ no\ \phi'\ \underline{with}\ nm' \in_? \Gamma$
```

```
$nm' \in_? (\iota :: \Gamma)$    $\mid no\ \phi'$ $\mid inj_2\ \phi = inj_2\ (next\ \{nm'\}\ \{\iota\}\ \phi)$
$nm' \in_? (\iota :: \Gamma)$    $\mid no\ \phi'$ $\mid inj_1\ \phi = inj_1\ (notFirst\ \phi'\ \phi)$
```

When the environment is empty, the identifier is not in the environment. We thus use inj_1 and need to construct a negation. This is a function, with an absurd pattern as first parameter, since none of the data constructors of $\_ \in \_$ can be applied. The other cases apply when the environment is not empty. Also note that patterns in Agda must be strictly linear: there may only be one introduction of an identifier. If there are multiple locations, the identifier must be prefixed with a dot.

In Section 9.6, we used the helper function *leftNil*. It is implemented by case distinction on its first argument. When α is empty, the requested property trivially holds. When α is not empty, normalization of $\alpha +\!\!+ \beta$ gives another constructor than $[\,]$, hence the absurd pattern.

```
$leftNil : (\alpha : Env) \to (\beta : Env) \to (\alpha +\!\!+ \beta \equiv [\,]) \to (\alpha \equiv [\,])$
$leftNil\ [\,]$          $\_\ refl = refl$
$leftNil\ (\_ :: \_)\ \_\ ()$
```

9.A.1 Absurd Rules

There may be productions for which no semantics for a given interface exists. For example, consider a grammar for a statically sized bit array.

```
grammar BitArray : ℕ → Set        -- statically sized bit array
  prod nil      : BitArray 0       -- empty bit array
  prod cons     : BitArray (suc n) -- non-empty bit array
    term?    n  : ℕ                -- length of the tail
    term     hd : Bool             -- bit at the head of the array
    nonterm  tl : BitArray n       -- tail of the bit array
```

We declare a synthesized attribute *head* that stands for the head bit of the array. We can extract this bit when the array is not empty. Hence, we state this requirement as inherited attribute.

```
itf BitArray n visit extract   -- Interface to extract the head
  inh prf   : n > 0            -- Proofs that the array is not empty
  syn head : Bool              -- Must return the head bit
```

If the array is empty, then we cannot give a value for *lhs.head*. Fortunately, the proof helps us out. If the array would be empty, then we would not be able to give the proof. Indeed, we can match with an absurd pattern against the proof, so that we do not have to give a definition for *lhs.head*.

```
datasem BitArray
  prod nil     lhs.head with lhs.prf   -- would be proof of suc 0 ⩽ 0
```

$$\begin{array}{lll} & \mid () & \text{-- not inhabited} \\ \textbf{prod}\ \mathit{cons} & \mathit{lhs.head} = \mathit{loc.hd} & \text{-- trivial} \end{array}$$

Under these conditions, a semantics for *nil* cannot be given.

When the interface has multiple attributes, or even multiple visits, then the above code would have to be duplicated for each synthesized attribute of the production, and each inherited attribute of the children. To prevent such code duplication, we introduce an absurd-rule. Its LHS p must be an absurd pattern that matches against the outcome of the RHS e.

$$r ::= \ \textbf{absurd}\ \ p = e \quad \text{-- absurd rule (with absurd } p\text{)}$$

For example, we can use it to match against *lhs.prf* in the *nil* production.

$$\begin{array}{l} \textbf{datasem}\ \mathit{BitArray}\ \textbf{prod}\ \mathit{nil} \\ \quad \textbf{absurd}\ \ () = \mathit{lhs.prf} \quad \text{-- would be proof of } \mathit{suc}\ 0 \leqslant 0 \end{array}$$

Attribute definitions and child declarations must be omitted if these would appear later than the absurd-rule in the rule order. The translation is relatively straight-forward.

$$[\![_{\mathsf{r}}\, \textbf{absurd}\, p = e]\!]_k \quad \leadsto \quad \underline{\mathit{with}}\ e\ \ \ldots \mid [\![p]\!] \quad \text{-- ends the current with-branch}$$

Since each with-branch ends in an absurd pattern, the continuation k is not needed. Since the follow-up rules on an absurd rule are not generated, we also demand that these are not specified in the first place.

These absurd-rules have consequences for the rule scheduling algorithm. We want the absurd-rules to be scheduled early, such that we can omit the rules that would follow. Also, we want their scheduling to be predictable, such that we know which rules we can and must omit. The scheduling algorithm consists of a number of phases.

- In the first phase, attributes are scheduled to visits of the interface. We assume that this step is performed manually, although it can be automated to a large extend, as is implemented in UUAGC [Löh et al., 1998].

- In the remaining phases, we can deal with each productions independently. For each production we construct a DAG that captures the dependencies between rules (Chapter 3). In this DAG, we identify the (indirect) predecessors of absurd-rules. These predecessors, combined with the absurd-rules, we schedule as early as possible. The partial order imposed by the DAG is turned into a total order by giving precedence first to absurd-rules, then evaluation rules, invoke rules, and finally child rules.

- In the final phase, we schedule the remaining rules as late as possible. Superfluous rules end up in the terminator visit.

If the DAG is non-cyclic, this algorithm properly schedules the rules. The DAG models the def/use dependencies of the rules. Each subsequent pass preserves these dependencies. Thus, the algorithm is sound. Also, if an absurd-rule r_1 could be scheduled earlier than a non-absurd rule r_2, then r_2 is not a predecessor of r_1. However, then r_1 would have been scheduled earlier as absurd-rules take precedence.

An absurd-rule thus forces the attributes its RHS refers to, to be scheduled as early as possible. It preferably does not have too complex dependencies (i.e. transitively speaking, only dependencies on inherited attributes and local attributes), so that it is clear when it is scheduled.

9.B Dependent Nonterminal Attribution

In Section 9.6, we showed how to deal with an attribute p that is only defined when another attribute q has a particular value v. The trick is to move p to a later visit than q, and add the invariant to p's visit that requires that it can only be invoked when q equals v. This invariant is expressed as additional attribute, which can then also be used in the definition of p. This is an example of a dependent attribution of nonterminals: depending on the value of q, the remaining attributes were either all present, or none were present. This approach can be generalized to allow a fixed number of different sets of attributes depending on the value of preceding attributes.

The responsibility for choosing a set of attributes can be given to the caller or the callee. The callee is the node that is a child of the caller. We divide responsibilities as follows. The caller invokes a visit on the callee, and is responsible for selecting one of the alternative interfaces that are offered by the callee. The callee is required to produce results for that choice. The callee can encode restrictions on the available choices for the parent as inherited attributes. The caller must provide values for the inherited attributes of the alternative interface it chooses. With this choice, both the caller and callee can impose demands. The caller imposes these demands by choosing an interface of the callee, and the callee imposes these demands through inherited attributes.

We change the syntax of interfaces to cater for *contexts*. A visit consists of a set of explicitly named contexts $\overline{z}$.

$i ::= \textbf{itf}\ I\ \overline{x} : \tau\ v$	-- with first visit v, params x, and signature τ
$v ::= \textbf{visit}\ x\ \overline{z}$	-- visit declaration with a set of contexts (many z)
$z ::= \textbf{context}\ x\ \textbf{inh}\ \overline{a}\ \textbf{syn}\ \overline{a}\ v$	-- context x for a visit

Notationally, layout becomes important. The contexts $\overline{z}$ must have the same indentation, and visits and contexts occurring inside a context must have a deeper indentation. As syntactic sugar, we assume that the context-keyword and name may be omitted if there is only one context for a visit. The syntax for the *terminator* visit is not needed anymore. It can be modeled as a visit with zero contexts. The names of contexts must be unique per interface. It allows us to distinguish contexts of different visits from each other, which is needed in $\mathrm{AG}_{\mathrm{DA}}$ because of its syntactical conveniences.

For example, we define two contexts for the *generate* visit of the earlier example. The context *errorfree* contains the *code* attribute, but it may only be invoked when errors are absent. The context *haserrors* provides a *pretty* attribute with a pretty print of the program. It does not pose restrictions on the *errors* attribute.

```
itf Source
  visit report   syn errors : Errs inh.finEnv
visit generate          -- a visit may consist of one or more contexts
  context errorfree     -- a context has a name
    inh noErrors : syn.errors ≡ [ ]
    syn code     : Target inh.finEnv
  context haserrors     -- a context may also contain subsequent visits
    syn pretty   : Doc
```

The callee must provide rules for each context. We split a semantics-block t for a visit up into a context u for each declared context of that visit in the corresponding interface. Such a context defines the next visit. For visits with more than one context, the caller must explicitly invoke them by means of an invoke rule.

```
t ::= visit x ū                      -- visit definition, with next visit t
u ::= context x r̄ t                  -- context with next visit t
r ::= invoke x₁ of c context x₂      -- modified invoke rule
```

Similarly to the notational conveniences above, we allow the context-keyword and name to be omitted if there is only one context declared for a visit.

For example, for production ◇, we choose contexts of the children depending on the context the visit itself is in. In this example, the context for the child is the same as the context of the parent. This is not a requirement.

```
datasem Source prod ◇
  lhs.errors = left.errors ++ right.errors            -- collect errors
  context errorfree                                   -- rules exclusive for errorfree
    invoke generate of left  context errorfree        -- explicit invoke
    invoke generate of right context errorfree        -- explicit invoke
    left.noErrors = leftNil  left.errors right.errors lhs.noErrors
    left.noErrors = rightNil left.errors right.errors lhs.noErrors
    lhs.code = left.code ◇ right.code
  context haserrors                                   -- rules exclusive for haserrors
    invoke generate of left  context haserrors        -- explicit invoke
    invoke generate of right context haserrors        -- explicit invoke
    lhs.pretty = left.pretty ⊕ right.pretty           -- collect pretty print
```

Ultimately, the choice for the context is made at the root. Either we make this choice external to the AG in terms of the generated coroutine, or use another AG extension [Middelkoop et al., 2010a]: *clauses*. A context may be split further into clauses. A clause may contain special match-rules, which may contain failing pattern matches. When a pattern match fails, execution backtracks to the next clause. The clauses are a means of case distinction on multiple rules at once. This is needed, in order to use different invoke-rules, depending on the values of an attribute.

```
itf Root visit compile  syn outcome : (Errs syn.finEnv) ⊎ (Target syn.finEnv)
datasem Root prod root                                -- some rules omitted
```

```
visit compile                                          -- clauses of visit/context
  clause emptyerrors                                   -- linear sequence of clauses
    match [] = top.errors                              -- possibly failing test
    top.noErrors = refl                                -- if match succeeds
    invoke generate of top context errorfree           -- take errorfree visit
    lhs.outcome = inj_2 top.code
  clause nonemptyerrors                                -- ¬([] ≡ top.errors)
    invoke generate of top context haserrors           -- take haserrors visit
    lhs.outcome = inj_1 top.errors
```

Clauses and match rules are confined to the visit/context they are defined in. The clauses must be exhaustive. Within these constraints, match-rules can be scheduled as usual. Match-rules are also scheduled as early as possible, similarly to absurd-rules (Section 9.A.1). There is a good reason to do so: we typically do not have explicit dependencies on a match rule, and its in general better to distinguish cases as soon as possible.

In the $\mathrm{AG}^{\mathrm{X}}_{\mathrm{DA}}$ translation, the visit function that corresponds to a visit has a parameter for each attribute of that visit. For the translation of contexts we give an additional, initial parameter to such a visit function. The value is a *handle* [Middelkoop et al., 2010b]: it describes what context we want, and what the type of that context is supposed to be. The handle can be considered a typed version of the control parameter in Kennedy and Warren [1976]. For example, for visit *generate*, there are two contexts, *errorfree* and *haserrors* respectively. For each context, we generate a type that specifies the types of the attributes (as described earlier):

```
T'Source'errorfree ...
T'Source'haserrors ...
```

Each constructor of the handle-type is indexed by one of these context types:

```
data H'Source'generate : (lhs_s gathEnv : Env) → (lhs_s errors : Errs lhs_s gathEnv) → Set → Set
  where
    errorfree : ∀ {lhs_s gathEnv} {lhs_s errors} →
      H'Source'translate lhs_s gathEnv lhs_s errors (T'Source'errorfree lhs_s gathEnv lhs_s errors)
    haserrors : ∀ {lhs_s gathEnv} {lhs_s errors} →
      H'Source'translate lhs_s gathEnv lhs_s errors (T'Source'haserrors lhs_s gathEnv lhs_s errors)
```

The visit function takes such a handle ($H'Generate\ \beta$) and returns β. The caller thus knows what β is, and the callee can find this out by pattern matching against the data constructors. The actual type of the visit function, and its implementation becomes:

```
T'Source'generate syn_a gathEnv syn_a errors
  = ∀ {β} → H'Source'syn_a gathEnv syn_a errors generate β → β
lhs_v generate : T'Source'generate lhs_s gathEnv lhs_s errors
lhs_v generate errorfree  = ...   -- translation for errorfree
lhs_v generate haserrors  = ...   -- translation for haserrors
```

The types in question seem rather complex, although this is mainly plumbing to pass attributes around on the type level. Fortunately, AGs alleviate us from writing such code by hand.

The presented mechanism opens up the possibility to have non-linear visit sequences. It also allows us to express interfaces for the ordered AGs as presented by Kennedy and Warren [1976]. Each context can represent an operation or query to be performed on the AST. For example, we could define an interface on the AST of types, such that one context computes the free variables, and another one applies a substitution to the type. The presented approach only permits branching. However, it may be possible to generalize the approach further and allow branches to merge and loop. Merging of branches could be of use for proofs of special cases that span multiple visits (Section 9.6). Also, the interfaces can be seen as *session types* for coroutines.

9.C Ideas Transferrable to AG Systems for Haskell

Some of the above ideas carry over to AG systems for Haskell, such as UUAG [Löh et al., 1998]. In a dependently typed AG, attributes can represent both values and types. In Haskell, there is a clear distinction between values and types. In an AG for Haskell, we can make an explicit distinction between attributes that represent types (and have a *kind* as type), and attributes that represent values. The type of a type attribute may not refer to other attributes. The type of a value attribute, however, may refer to a type attribute. Type attributes correspond to quantification. An inherited type attribute corresponds to universal quantification, since the caller can choose its instantiation. A synthesized type attribute corresponds to existential quantification. The callee can choose its type, but the caller cannot make an assumption about it. This mechanism allows us to deal with polymorphism in interfaces.

Currently, UUAG only supports kind star data types. We showed how to deal with parameterized data types, and GADT-style data constructors. These extensions allow AGs to be written for data types with a stronger typing discipline. Also, we can deal with class constraints in constructors by introducing explicit wrappers for dictionaries that we can store as additional fields, e.g.:

```
data DictEq :: * → * where
  DictEq :: Eq a ⇒ DictEq a
```

With such extensions, we can handle even non-regular data types, as long as nonterminals have an outermost type construct described by a grammar declaration.

The handles of Appendix 9.B are conventional GADTs, hence the context-idea can be implemented in Haskell.

10 AGs on Graphs

This chapter shows how to express attribute grammars on graph structures using the extensions that we presented in the previous chapters. Moreover, we show how these extensions form an alternative to Reference Attributed Grammars (RAGs). Unlike RAGs, these extensions are compatible with ordered attribute grammars. In fact, these extensions rely on a static analysis of attribute dependencies. Finally, this chapter can be seen as a showcase of how the extensions fit together in a wider context than type inference.

10.1 Introduction

Directed graphs often occur as intermediate representation in compilers. Graph traversals are inherently more difficult than tree traversals, because nodes can have multiple predecessors, and graphs can be cyclic. Yet, the concept of inherited and synthesized attributes appears attractive to model fixpoint computations on control flow graphs and dependency graphs, as shown by Reference Attribute Grammars (RAGs).

In RAGs [Magnusson and Hedin, 2007], references to nodes are first class. A reference to a node provides access to synthesized attributes of that node. The graph is formed by taking a children of a node as the node's successors, and the node's optional parent and referenced nodes as predecessors. There is a subtle different between a parent and a referenced node: inherited attributes are accessed via the reference to the parent, whereas synthesized attributes are accessed via references to other nodes. Attributes that are accessed via a reference are called reference attributes. In the presence of reference attributes, it is not obvious that all attributes are well-defined.

In a tree, if an inherited attribute of a node *n* depends on an attribute of a subtree, then it also depends on a synthesized attribute of *n*. Similarly, if a synthesized attribute of a node *n* depends on an attribute of a parent, then it also depends on an inherited attribute of *n*. These properties allows us to abstract the dependencies of a subtree of a production to the dependency graph of the associated nonterminal. These properties thus allows statical analysis of attribute dependencies, and allow us to compute attributes using regular treewalks.

In general, the above properties do not hold for graphs. For example, attributes of siblings can be accessed via a reference without having dependencies on an attribute of a common ancestor. Moreover, cyclic graphs may cause attributes to be accidentally or purposefully cyclic, and we may need more control over fixpoint computations in the latter case. Consequently, well-definedness of RAGs cannot be determined statically, which complicates reasoning with attributes.

In this chapter, we show with examples how techniques from Chapter 5) provide an alternative to reference attributes in ordered attribute grammars. Effectively, reference attributes are a means to observe attribute values from other locations in the tree and produce attribute values that are observable at other locations. We obtain a similar behavior by shuffling actual children (instead of references) around. When we detach a child at one location, and attach it at another location, then we may define and access some of its attributes at that location. Detached children are first class functional values that can be passed around via attributes, and may be duplicated.

However, this raises the question what the state a child is (i.e. what attributes of it are computed) when we detach or attach it. We showed in Chapter 5 that when we partition the evaluation of attributes as a sequence of visits, that each visit to a child represents a state transformation of that child. The interface of a nonterminal is a static description of the intermediate states of the child from the perspective of the parent. This mechanism allows us to detach and attach children, and still statically guarantee that their attributes are well-defined. The catch is that when we attach a child, we are actually required to provide a value value that represents the child to attach. Typically, we look such a value up in an environment, which implies that the value must be present in that environment.

The mechanism of shuffling children around still allows us to reason with attributes in a conventional purely functional way. We demonstrate this mechanism with a control flow example in Section 10.2 and an example based on symbol tables in Section 10.3.

10.2 Analyses on Control Flow Graphs

We present a relatively simple control flow analysis with AGs. This example is based on experiences with transformations of bytecode for the AVM2 virtual machine. AVM2 is the runtime of ActionScript version 3 programs. In bytecode, the body of an ActionScript method is a sequence of AVM2 instructions. The execution of an instruction may have an effect on the stack. For example, during a call instruction the arguments of the function are popped off the stack and the result of the method is pushed on the stack. AVM2 reserves stack space upon entry of the method, and requires a priori knowledge of the maximum size of the stack after any instruction, given an empty stack upon method entry. In this section, we show an analysis with AGs that computes this number as an attribute *maxStack* of a sequence of instructions.

10.2.1 Abstract Syntax Tree of Byte Code

Figure 10.1 shows an example of the AST of a compiled ActionScript method that represents a repeated invocation of some method *m* that takes an integer as argument and returns an integer as result. The following is a simplified representation of such ASTs:

```
grammar Method                                -- nonterminal represents methods bodies
   prod Body      nonterm instrs : Instrs     -- a method is a cons-list of instructions
grammar Instrs                                -- nonterminal represents instructions
   prod Nil                                                 -- empty sequence
   prod Cons      nonterm hd : Instr   nonterm tl : Instrs  -- hd followed by tl
grammar Instr                                 -- nonterminal represents instructions
   prod Nop                                   -- no-op
   prod Dup                                   -- duplicates top of stack
```

prod *Push*	**term** *val*	:: *Int*	-- pushes *val* on top of stack
prod *Call*	**term** *method*	:: *Ident*	-- pops args, calls *method*, pushes result
prod *Jump*	**term** *index*	:: *Int*	-- execution continues with *index*
prod *JmpZero*	**term** *index*	:: *Int*	-- pops top, continues with *index* if zero

By default, AVM2 executes instructions of a method in the order of appearance. However, after a (conditional) branch instruction, such as *Jump* and *JmpZero*, the execution continues with the instruction as specified by the branch instruction (if the condition is met).

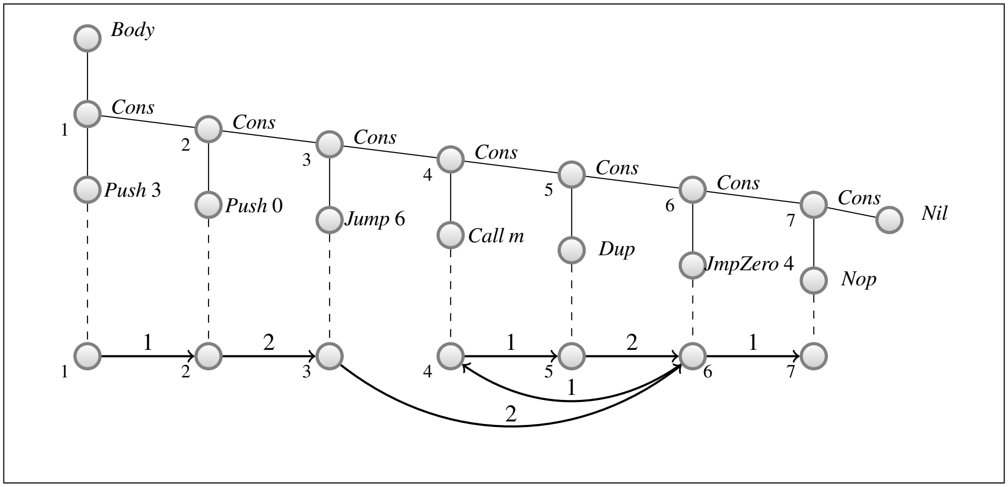


Figure 10.1: Bytecode AST with CFG.

For such a sequence of instructions, we define our analysis as a synthesized attribute *maxStack* with the following semantics. In passing, we use an inherited attribute *env* that specifies the number of arguments and results for each method, and an inherited attribute *ind* for the sequential numbering of the instructions:

```
itf Instrs
    inh env        :: Map Ident (Int, Int)    -- maps method to number of args and results
    inh ind        :: Int                     -- begin index of instruction sequence
    syn maxStack   :: Int                     -- maximum after-stack of instr sequence
sem Method
    prod Body    instrs.env = ...             -- obtained from method declarations
                 instrs.ind = 1               -- index of the first instruction
sem Instrs
    prod Nil     lhs.maxStack = 0
    prod Cons    lhs.maxStack = hd.maxStack `max` tl.maxStack
                 tl.ind       = 1 + lhs.ind   -- index of next instruction
                 hd.env       = lhs.env       -- pass down env
```

For each instruction *i*, we define the size of the stack *i.after* in terms of the size of the stack *i.before*:

```
itf Instr
  inh env       :: Map Ident (Int,Int)       -- method to number of args and results
  inh before    :: Int                       -- size of stack before this instruction
  syn after     :: Int                       -- size of stack after executing this instr
  syn maxStack  :: Int                       -- max size of stack after this instr
sem Instr
  lhs.after     = lhs.before + loc.effect    -- loc.effect different for each instr
  lhs.maxStack  = ...                        -- defined later

  prod Nop Jump   loc.effect = 0             -- exec has no effect on stack size
  prod Dup Push   loc.effect = +1            -- exec adds one to the stack size
  prod JmpZero    loc.effect = −1            -- exec pops top of the stack
  prod Call       loc.effect = loc.results − loc.args
                  (loc.args, loc.results) = lookup method lhs.env
```

So far, the analysis appears to be a straightforward attribute grammar. However, this is not the case when we try to define *hd.before* in production *Cons*, and *lhs.maxStack* in productions of *Instr*. The size *i.before* of an instruction *i* is equal to *j.after* for any last instruction *j* that is executed prior to executing *i*. Due to branching instructions, there may be several *j.after* sizes for *i.before* (which all need to be the same), and *i* may be before or after *j* in the instruction sequence. Such definitions may be cyclic, for example when $i \equiv j$ (e.g. a loop with empty body). It is therefore not obvious how to define these attributes in terms of the cons-list of instructions.

10.2.2 Control Flow Graph

As solution, we construct a static Control Flow Graph (CFG), and define the remaining part of the analysis in terms of this graph. A *control flow graph* is a directed graph where each instruction uniquely corresponds with a vertex. There is an edge from a vertex *i* to vertex *j* if the execution may continue with *j* after executing *i*. Figure 10.1 shows an example of a CFG derived from an exemplary AST of a method body. Note that there is no edge between vertex 3 and vertex 4 because of the jump of instruction 3. Also, there is a *back edge* from vertex 6 to vertex 4 due to the conditional branching of instruction 6. The analysis labels each edge with the size of the stack after executing the source vertex of the edge and prior to the execution of the target vertex of the edge.

The incoming edges of a vertex must be labeled with the same stack size, similarly for the outgoing edges of a vertex. Instead of labeling the edges with the stack size, we can thus label the vertices with the stack size. The analysis then boils down to the following depth-first traversal. We start with a stack size of 0 at vertex 0. A vertex that is visited via a predecessor with stack size *s*, has a stack size of $s + loc.effect$, where *loc.effect* is the effect on the stack of the associated instruction.

There are multiple ways to construct such graphs with attribute grammars. With RAGs or cyclic AGs, we can construct the graph implicitly. For that, we introduce an attribute that contains the *after* value for each instruction, and define a list of predecessors for each instruction, so that we define the *after* attribute in terms of the *after* values of the predecessors. However, these AG extensions use a fixpoint computation to compute the attribute values, which may actually lead to nontermination when the join operation is naively defined as *max*. The fixpoint computation is also more expensive than a single depth-first traversal.

As an alternative, we explicitly construct the graph. For that, we introduce attributes that specify where instructions branch to. An instruction may be followed up with the next in the sequence, unless attribute *hd.isJump* is *True*. Moreover, an instruction may be followed up by *hd.branch* if its value is not *Nothing*:

```
itf Instr
  syn isJump :: Bool          -- whether it is the Jump instruction
  syn branch :: Maybe Int     -- non-sequential label to branch to
sem Instr
  lhs.isJump  = False         -- defaults for isJump and branch
  lhs.branch  = Nothing       -- Nothing: no non-sequential branch
  prod Jump             lhs.isJump = True
  prod Jump JmpZero     lhs.branch = Just index    -- branches to instruction index
```

Furthermore, we introduce attributes *vertices* and *edges*. The index of each instruction is the unique identification of the associated vertex. At the top of the instruction sequence, we use a graph library to construct a graph from the vertices and edges, and traverse the graph depth-first to obtain the resulting vertices. Attribute *outcome* distributes these vertices. We explain below how we actually represent the vertices:

```
itf Instrs
  syn vertices :: Map Int ...    -- vertices of the sequence of instrs
  syn edges    :: [(Int, Int)]   -- edges of the sequence of instrs
  inh outcome  :: Map Int ...    -- contains result vertices of the instrs
sem Instrs
  prod Nil    lhs.vertices   = ∅
              lhs.edges      = [ ]
  prod Cons   lhs.vertices   = insert lhs.index loc.vertex tl.nodes
              lhs.edges      = [(lhs.index, s) | s ← loc.succs] ++ tl.edges
              loc.succs      = loc.seqts ++ maybeToList hd.branch
              loc.seqts      = if hd.isJump then [ ] else [lhs.index + 1]
              loc.vertexOut  = lookup lhs.index lhs.outcome
              loc.vertexIn   = ...   -- definition of vertex explained below
sem Method prod Body
  instrs.outcome = dfvisit 0 instrs.vertices instrs.edges
```

As a first approach, we take the *loc.effect* value of *hd* for *loc.vertexIn*, and replace in *dfvisit* a vertex labelled with effect e with $s + e$ where s is the stack value of the predecessor of the vertex in the traversal, or 0 if the vertex does not have a predecessor. However, this approach has the disadvantage that we manually pack and unpack context in the graph. For example, we may need to additionally pack compiler options, and additionally unpack error messages aside from the stack size. Also, the logic for each visit belongs with the associated instruction, not with the top of the instruction sequence.

Therefore, we take a different approach. *The instruction node itself becomes the vertex in the graph.* The following is the interface for *Instr* again, but this time with the attributes declared explicitly in three visits:

```
itf Instr
  visit analysis
    inh env      :: Map Ident (Int,Int)   -- method to number of args and results
    syn isJump   :: Bool                  -- whether it is the Jump instruction
    syn branch   :: Maybe Int             -- non-sequential label to branch to
  visit graph
    inh before   :: Int                   -- size of stack before this instruction
    syn after    :: Int                   -- size of stack after executing this instr
  visit outcome
    syn maxStack :: Int                   -- max size of stack after this instr
```

The interface allows us to specify that the *graph* visit is not performed by *hd*, but performed by some external means. In this example, the *graph* visit is performed by the graph traversal. We detach *hd* from the *Cons* production as attribute *loc.vertexIn* when it reached the *graph* state, and attach it again when it is delivered in the *outcome* state as attribute *loc.vertexOut*:

```
sem Instrs prod Cons
  loc.vertexIn = detach graph of hd            -- detach hd in state graph
  attach outcome of hd : Instr = loc.vertexOut -- attach hd in state outcome
```

As a minor detail, the *Cons* node does not perform the *graph* visit, therefore it may not access *hd.after*. However, since the *graph* visit for the instruction node was performed, we may access these attributes at this node. Thus, we expose this attribute in a later visit as *maxStack*:

```
sem Instr   lhs.maxStack = syn.lhs.after   -- syn.lhs.after computed by prev visit
```

With this mechanism, we thus specify the packing boilerplate (via detach) and unpacking boilerplate (via attach) of the node as vertex in the graph once. For the definition of the semantics of the node we conveniently have access to its attributes, and may use the attributes introduced by a visit that is visited externally in subsequent visits.

10.2.3 Discussion

The depth-first traversal *dfvisit* invokes the *graph* visit (e.g. through the associated wrapper function) and supplies a value for the *before* attribute. The result of the invocation is the value of the *after* attribute and the node in the *outcome* state. The former value forms the input for the visits to the successors of the node, and the latter value is the transformed value of the vertex. The following definition of *dfvisit* serves as illustration:

```
dfvisit :: α → Map Int (α → (α, β)) → [(Int, Int)] → Map Int β
dfvisit initial vertices edges = visMany initial empty sources where
  sources      = keys vertices `difference` concatMap snd edges
  visMany inp  = foldl (visSingle inp)
  visSingle inp results node
    | member node results = results
    | otherwise           = let (out, next) = lookup node vertices inp
                                succs       = [t | (s, t) ← edges, s ≡ node]
                            in visMany out (insert node next results) succs
```

The attributes of the *graph* visit form the interface with the graph traversal. Extra attributes can be defined to tune the traversal. For example, in case of a fixpoint traversal, a synthesized Boolean attribute may specify that the node's state did not change. With this mechanism, we thus separated the traversal strategy from the semantics of the node.

When a node is attached again, it is essential that it is an appropriate node. We do not statically guarantee that this is the case. A detached node may be duplicated, swapped or lost. The latter case causes a runtime error when an attempt is made to attach the node, as it is absent in the *outcome* map. The former two cases may be intended, although we can formulate a runtime check for them. Chapter 9 shows how to statically enforce such invariants.

10.3 Subtrees in Symbol Tables

Abstraction in programming languages entails that terms of the language can be given a name (thus defining or declaring that name), which may then be used at other locations in the program by referring to that name. Consequently, in an attribute grammar, at a node that uses such a name, we refer to properties of the subtree that introduces this name. To access such properties, we usually pass these properties around using attributes that represent *symbol tables* (maps from name to some value).

Symbol tables are a technique to explicitly transfer properties of a defining node to its use nodes. This technique involves boilerplate code to wrap properties of the defining nodes in record-like data structures, and unwrap these properties at use nodes. We present a technique to reduce this boilerplate code by transferring the tree of the defining node via symbol tables to its use nodes.

10.3.1 Abstract Syntax of an ActionScript Module

We present our technique with an example of an analysis that requires information from a class declaration to deal with a call to a method of that class. Similar to the example of Section 10.2, this example is based on our experiences with the transformation of ActionScript bytecode.

In this setting, a module consist of a sequence of declarations, where a declaration is either a class declaration or a method body. A class declaration specifies the traits of an object that is an instance of that class. A trait is either a field which stores some value, or a method which can be called. A method body may be bound to a compatible method-trait of an object. A method body consists of a sequence of instructions. Relevant for this example is the method call instruction. The following is a simplification of the relevant abstract syntax:

```
grammar Module
   prod Module    nonterm decls : Decls
grammar Decl
   prod Class     nonterm decl : ClassDecl
   prod Body      nonterm decl : MethodBody
grammar ClassDecl
   prod Class     term nm :: Name      nonterm traits : Traits
grammar Traits
   prod Nil
   prod Cons      nonterm hd : Trait   nonterm tl : Traits
grammar Trait
   prod Field     ...
   prod Method    nonterm decl : MethodDecl
grammar MethodDecl
   prod Method    term nm :: Name      nonterm params : Params
grammar Params
   prod Nil
   prod Param     term tp :: Type      nonterm tl : Params
grammar Instr
   prod Call      term cl :: Name      term trait :: Name
```

We omitted some of the intermediate nonterminals (e.g. *Decls* and *MethodBody*) that are not relevant for this example.

A method call specifies which method of which class to call. When the instruction is executed, an object that is an instance of this class must be on top of the stack, as well as arguments of the right types. For typical compilation tasks for the call instruction (e.g. the analysis of Section 10.2) we need several properties of the relevant class, the relevant trait, and the relevant parameters. For such properties, the structure in the symbol table is typically an abstraction that has a similar structure as the class declaration subtree itself. Figure 10.2 depicts this situation. At the call instruction node, we obtain through symbol tables an abstraction of the declaration subtree (larger triangle), and extract from that an abstraction of the appropriate trait subtree (smaller triangle).

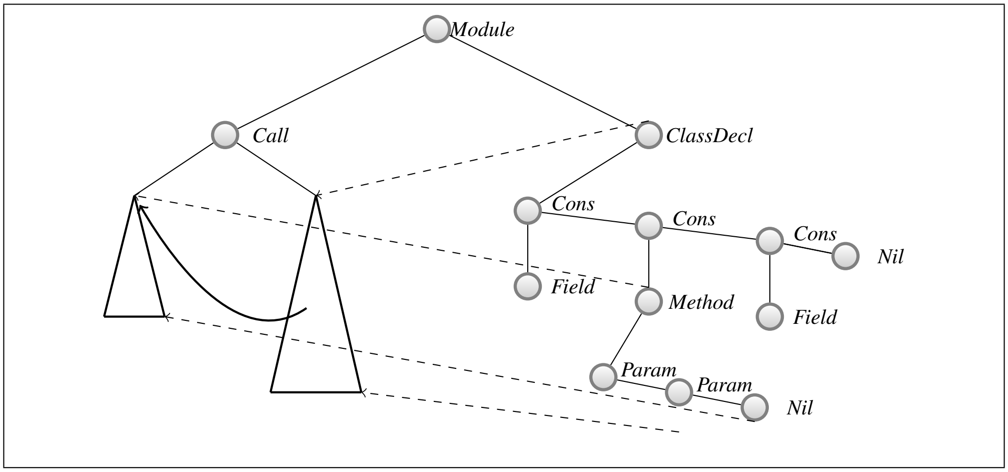


Figure 10.2: AST with a method call that needs information from class declarations.

10.3.2 Distribution of Subtrees

Instead of creating special data structures to represent the abstractions in the symbol table, we show how to use the decorated declaration subtree itself as abstraction. For that purpose, we partition attributes for the declarations in visits: a *def* visit with attributes for the definition node, and visits *use*$_1$ and *use*$_2$ with attributes for the use nodes. These attributes represent typical properties of class and method declarations:

```
itf ClassDecl
  visit def                                 -- more attributes of def omitted
    syn nm        :: Name                   -- name of the class
  visit use₁
    inh moduleNm  :: Name                   -- context in which the class is used
    syn visible   :: Bool                   -- whether or not the class is visible
  visit use₂
    inh methodNm  :: Name                   -- name of method to lookup
    syn method    :: I_MethodDecl_use       -- decl associated with methodNm
itf MethodDecl
  visit def                                 -- more attributes of def omitted
    inh index     :: Name                   -- index of trait assigned by parent
    syn nm        :: Name                   -- name of the method
  visit use
    syn slot      :: Int                    -- index of the trait in the object
    syn numParams :: Int                    -- number of parameters
```

The type *I_ClassDecl_use*$_1$ is the type of a detached *ClassDecl* tree in the the *use*$_1$ state. When we attach a node in this state, we may use the *visible* attribute when we provide a value for the *moduleNm* attribute. Also, when we provide the name of a method as attribute *methodNm*, we obtain the associated detached method as attribute *method*. The type *I_MethodDecl_use* is the type of a detached *MethodDecl* that can be attached starting with visit *use*.

We collect detached nodes with the above interfaces in several symbol tables.

```
itf Traits Trait
   syn gath :: Map Name I_MethodDecl_use     -- collects method traits
itf Decls Decl
   syn gath :: Map Name I_ClassDecl_use_1    -- collects class declarations
sem Trait prod Method
   loc.tree  = detach use of decl            -- detach tree in the use state
   lhs.gath = singleton decl.nm loc.tree     -- collected detached tree
sem Decl prod Class
   loc.tree  = detach use_1 of decl          -- detach tree in the use state
   lhs.gath = singleton decl.nm loc.tree     -- collected detached tree
itf Decls Decl MethodBody Instrs Instr
   inh dist  :: Map Name I_ClassDecl_use_1   -- distribution of symbol table
sem Module prod Module
   decls.dist = decl.gath                    -- pass down gathered classes
sem Instr prod Call                          -- attach the trees
   attach use_1 of c  : ClassDecl    = lookup cl lhs.dist
   attach use   of m : MethodDecl = c.method
```

Aside from the detaching and attaching of the nodes, the collection and distribution of the nodes is standard attribute grammar practice.

We define the properties that are needed at the use node as synthesized attributes of the definition node. For the definition of these attributes we may use attributes that are in scope of the definition node:

```
sem ClassDecl prod Class
   lhs.nm           = nm                 -- pass name up (visit def)
   lhs.visible      = ¬ private ∨ ...lhs.moduleNm ...
   lhs.method       = lookup lhs.methodNm traits.gath
sem MethodDecl prod Method
   lhs.nm           = nm                 -- pass name up (visit def)
   lhs.slot         = lhs.index          -- pass the assigned index up (visit use)
   lhs.numParams = params.count     -- pass the number of params up (visit use)
```

A particular advantage of this approach is that by adding more synthesized attributes, we expose properties of the definition node without additional boilerplate code to transfer these properties to the use node. At the use node we simply refer to these attributes:

```
sem Instr prod Call
   c.moduleNm = lhs.moduleNm
   c.methodNm = trait
```

$$\begin{aligned} &\mathit{lhs.errors} &&= \textbf{if}\ \mathit{c.visible}\ \textbf{then}\ [\,]\ \textbf{else}\ [\mathit{ClassHiddenError}] \\ &\mathit{lhs.code} &&= \ldots \mathit{t.slot} \ldots \mathit{t.numParams} \ldots \end{aligned}$$

Another advantage is the separation of concerns. At the use node, we specify in which module the use node occurs, and at the definition node we determine if the class declaration is visible at the use node. Similarly, at the use node, we just specify the method in which we are interested and the definition side provides the relevant subtree, and e.g. is responsible for producing error messages.

10.3.3 Discussion

Accessing information from other locations in the tree is a common pattern in compiler implementations, and we typically use symbol tables to transfer the information from one location to the other. However, when the data stored in the symbol table has a structure that mimics the AST itself, tedious boilerplate is required to store and retrieve information from the symbol table. We showed how to transport subtrees using symbol tables, which can then be used to access attributes of the remote location.

RAGs permit access to synthesized attributes to nodes at arbitrary locations in the tree through references to these nodes. In contrast to RAGs, we permit that use-nodes also provide values for inherited attributes, such that the use-node is effectively a client and the definition-node a server. We can thus provide values for synthesized attributes at the definition side, while taking context of the use node into account.

In fact, the interfaces can be seen as a specification of queries to a subtree. The caller specifies the query as values of the inherited attributes, and the subtree responds with values for the synthesized attributes. We designed this example such that every use node queries the definition side in the same way. This is generally not the case. For example, for field assignment instructions we are interested in the field traits of a class, and not the method traits. In Section 9.B we show how to specify alternative interfaces for a nonterminal, which can be used to query a nonterminal in different ways.

A subtree may be duplicated many times. Since we treat detached trees as purely functional values, attached trees are independent of each other. Attributes that do not depend on context of the use node can be computed before detaching the tree at the definition node and are only evaluated once. Attributes that are computed in the visits performed by the use node, however, are thus computed once per use node. If such an attribute has a non-trivial computation, then this is a candidate for memoization, since there are typically not many different contexts in a program.

10.4 Related Work: Door Attribute Grammars

Door Attribute Grammars (DOGs) [Hedin, 1994] share commonalities with our approach to reference attributes. A DOG introduces a different kind of AST node, called a *door node*. Door nodes may not have children, and carry no syntactic content. References to door nodes may be passed as attribute values, and may be attached as a child. These nodes thus serve as interface to attributes of another door node at another location in the tree. Inherited attributes at the source location become synthesized attributes at reference location.

An essential difference with our approach is that attributes of a DOG are evaluated on demand. In comparison to RAGs, DOGs must specify precisely which attributes are exchanged between two locations in the tree.

10.5 Conclusion

We showed how the explicit visits of Chapter 5 can be used to shuffle children around. This mechanism forms an alternative to reference attributes, and can be used to abstract from common patterns in compiler implementations related to symbol tables and fixpoint computations over dependency or flow graphs.

These abstractions pay off when implementing mature compilers for larger languages. For such languages, context information plays an important role. For example, context information in the form of position information, error messages, several environments, and runtime options such as iteration limits. To have such context available automatically via attributes saves the manual packing and unpacking of context information into graph representations, which makes it easier to extend such solving algorithms with more context information.

11 Conclusion

This thesis presented several extensions to attribute grammars. With these extensions, complex type inference algorithms can be expressed. When viewed from a high level perspective, this thesis explored notation that allows units of evaluation in an AG to be made explicit, to be annotated, and to be controlled. More concretely, we regard type inference as the simultaneous construction of a derivation tree and the evaluation of the tree's attributes. In an inference algorithm, attribute evaluation and tree construction take turns. The construction of a derivation tree thus introduces the notion of phasing. Certain attributes must be evaluated to decide on the structure of the tree. We provided several examples, in particular in Chapter 5.

With our extensions, we organize the evaluation of attributes in phases, which makes explicit in what state the tree is before and after the evaluation of a phase. We showed how to exploit this additional information to encode fixpoint computations and search algorithms. Although we focussed on type inference, our extensions are actually making it possible to describe complex tree walking automata.

11.1 Addressed Challenges

We briefly restate the challenges as mentioned in the instruction. A declarative specification of a type system is used for formal reasoning and explanation. An inference algorithm describes how to infer admissible types for a program. The former is therefore an implementation of the latter. To show that an inference algorithm is indeed an implementation, it is desirable to have a proof that it is consistent with the type system.

As discussed in Section 1.1, it is a non-trivial exercise to construct an inference algorithm for languages such as Haskell, which integrate several cross-cutting type systems. In this setting, a formal specification (if it exists at all) and the implementation are complex, which makes it hard to keep both consistent, let alone proof that this is the case. Moreover, such an implementation changes continuously to support more language features or to use different implementation techniques, thus it is likely to be inconsistent. The apparent inconsistency leaves a big gap between theory and practice. With this thesis, we worked towards a solution where the gap is closed by specifying a formal type systems that contain sufficient algorithmic details to be executable.

A question that arose is what execution model to associate with type rules. Despite the absence of a general inference algorithm that fits any type system (Section 1.2.4), most inference algorithms are a complex composition of standard inference techniques. An executable specification thus describes how to tailor these algorithms to the type system. In this thesis, we zoomed in on the description of such underlying algorithms.

Previous work on the language Ruler posed an initial solution. Ruler is a language in which syntax-directed, algorithmic type rules can be described, which in combination with some additional annotations can be translated automatically to AGs (Section 1.4). Type rules coincide with productions, and metavariables with attributes. The inference algorithm is thus expressed as the evaluation of AGs.

For a declaratively specified type system, type inference for declarative aspects boils down to inferring the structure of the derivation tree and inferring instantiations for non-functionally constrained meta variables. Higher-order AGs withstanding, AGs assume an apriori fixed tree, and require functional definitions for attributes. Therefore, an inference algorithm for declarative type rules cannot be obtained via a straightforward translation to attribute grammars. We addressed this issue in this thesis by extending the AG language to cater for several inference strategies, and mechanism to combine these strategies.

11.2 Solutions

As initial setting, we showed that we can model an AG on type derivation trees instead of parse trees. To deal with non-functional attributes such as types, we used a conventional unification-based strategy with an additional threaded substitution attribute (Section 1.3.11), and expressed unifications in the AG in a declarative manner using higher-order children (Section 1.3.12). The substitution in combination with placeholders in types makes the relations on types functional. Moreover, the implementation of the higher-order child expresses the standard unification algorithm, and the threading of the substitution its algorithmic coordination, since the threading defines the relative order of the unifications and other operations that depend on the substitution. To this setting, we made various improvements which we describe in the remainder of this section.

The explicit threading of the substitution is a tedious job. Alternatively, we can regard unification as an operation with side effects that affects the substitution. In Chapter 3, we showed how to make the visit order explicit and use this order to declaratively specify the relative order of the operations with side effects in addition to the usual order constraints induced by attribute dependencies. To retain the desired referential transparency of attributes in the programming model, side effects may only be used in the construction of higher-order children. In particular, this approach allows AGs to be integrated in a compiler that uses inference monads.

The relative order of operations that provide fresh placeholders and perform unification is largely irrelevant. In Chapter 4, we presented commutable rules for threaded attributes to both model side effects and to relax the order induced by attribute dependencies. In Chapter 4, we abstracted from visits to phases, where a phase corresponds to one or more inferred visits, and allowed us to decouple the actual AG evaluation algorithm. Visits provide a fine-grained model to describe aspects of the evaluation of AGs, whereas with phases we can specify properties of larger chunks of evaluation. We also showed how to encode the Kennedy-Warren AG evaluation algorithm in a strongly-typed functional language.

We exploited the notion of a visit to express typical inference algorithms. In Chapter 5, we presented how to express fixpoint iteration by iterating visits. An invocation of a visit on a subtree specifies how to compute the visit's inherited attributes from its synthesized attributes of the previous iteration. In contrast to conventional fixpoint evaluators for AGs, the notion of visits allows us to compute stop conditions as synthesized attribute, and have visit-local chained attributes that retain computed values of previous visits.

When an attribute contains an unconstrained placeholder, residuation is a strategy that defers a dependent computation until the placeholder is sufficiently constrained. In Chapter 5, we presented an example that implements the residuation strategy by decoupling a child at one location in the tree and provide input at another location in the tree. This mechanism also provides an integration with constraints (Section 5.2.7).

In Chapter 5, we also presented how to infer the derivation tree as a function of the inherited attributes. In a case study [Middelkoop, 2011a], we illustrated the need for search strategies to infer the structure of derivation trees when the structure of the derivation tree is not functionally defined. Chapter 7 showed how to encode such search strategies.

Finally, our work provides solutions in other contexts than type inference. We illustrated this in the extended edition of this thesis [Middelkoop, 2011b], where we applied our work to graphs. Since many analyses in compilers are based on control-flow or data-flow graphs, we showed how to associate a semantics with each node of such a graph with an AG, while explicitly specify visits to these nodes with a traversal algorithm. Moreover, this approach allows the encoding of reference attributes in an ordered AG.

In Chapter 9, we applied our work to dependently-typed languages. With dependently-typed attributes, invariants between attributes can be expressed, and proven to hold, which allows formal reasoning with attribute grammars. Moreover, the type attributes provide a mechanism for the universal and existential quantification of the type of a semantic functions.

11.3 Remarks

We put great effort in ensuring that our extensions retain the ease of composition as offered by AGs so that attributes and rules can still be defined in an aspect-wise and order-independent fashion. Also, our ideas are conservative extensions of AGs. A conventional AG can be expressed straightforwardly, which allows features of RulerCore to be retrofitted on an implementation using conventional AGs.

However, to exploit the visit order, we need to specify in which order the children are visited. For conventional AGs, the traversal over the AST and order of evaluation of rules does not need to be specified, which has the advantage that it is no concern for the programmer. This is only a small price to pay. Such orders are declaratively specified (Chapter 3), and only where needed.

11.4 Implementations

We implemented several prototypes to experiment with the extensions presented in this thesis. We validated that our extensions have reasonably efficient implementations. In particular, the ideas are implementable in Haskell. Our prototypes depend on language features such as lazy evaluation, monads and GADTs. However, the underlying ideas themselves are implementable in mainstream languages and other AG systems.

To our attribute grammar system UUAG, we added higher-order children (Section 1.3.7) and stepwise evaluation (Chapter 7). In essence, we added only features that do not conflict with the conventional notion of AGs. Also, we implemented several features to support the use of AGs in large compiler implementations. In particular, we organized the code generated by UUAG such that separate compilation, debugging, and profiling is possible.

In the tool `ruler-core`, we implemented the programming model with explicit visits (Chapter 3). We experimented with clauses, fixpoint iteration, and other features that benefit from the notion of a visit (Chapter 5). As further case study, `ruler-core` was used in a master project to implement the inference algorithm of the HML type system [Leijen, 2009].

In the tool `ruler-interpreter` we implemented the operational semantics as outlined in Chapter 4. The interpreter provides custom judgment syntax, aspect-weaving of type rules, and a built-in efficient unification mechanism. The simplicity of an interpreter facilitated rapid prototyping with some of RulerCore's features.

11.5 Future Work

We implemented and validated our ideas with several prototype implementations. We provided powerful building blocks for the description of inference algorithms with AGs, and the description of patterns that often occur in these contexts. However, to fully exploit our techniques, an integrated implementation of all extensions is needed.

The features that we implemented in UUAG were used to improve the UHC implementation. Moreover, we used UHC as motivation to investigate AG extensions using RulerCore for prototyping purposes. We claim that we provide sufficient expressive power to implement unification and context reduction more concisely in UHC, but to actually do so remains as future work.

We provide the algorithmic underpinning of inference algorithms for AGs. This takes us one step closer to our ultimate goal to derive type inference algorithms from type system specifications. However, the remaining challenges as mentioned in Section 1.5 need to be addressed as well, such as special syntax and first-class abstractions for AGs. Moreover, visits are a prominent component in our current work. A direction of future work is to *infer* properties of visits from a more abstract specification.

A question that remains open is how we can formally prove and ensure properties of the implementations that we generate. We address this topic briefly in Chapter 9. A direction of future work is to generate boilerplate code to support typical proofs.

This thesis provides a core language for inference algorithm descriptions, which paves the way for high-level abstractions of inference algorithms, thus facilitates more complex language implementations, and ultimately leads to software of higher quality.

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Samenvatting

Computers zijn niet meer uit onze maatschappij weg te denken. Computerprogramma's worden steeds omvangrijker. De keerzijde is dat deze programma's meer tijd kosten om te maken en te testen, en bovendien ook meer fouten bevatten. Helaas ervaren we regelmatig de consequenties ervan. Autofabrikanten roepen bijvoorbeeld auto's terug om een software-update door te voeren, omdat door fouten in de programmering de auto's op hol konden slaan. Ook fouten in web-browsers worden door duistere figuren uitgebuit om zogenaamde *keyloggers* op computers te installeren die toetsaanslagen afluisteren om uiteindelijk bankrekeningen te plunderen. Oplossingen voor dergelijke problemen kunnen deels in programmeertalen gezocht worden. Onderzoek naar programmeertalen helpt om zowel de onderhoudskosten en de kwaliteit van software te verbeteren.

Met geavanceerde programmeertalen is het mogelijk om computerprogramma's van hoge kwaliteit te maken. Daarvoor is een belangrijk stuk gereedschap van belang: de *compiler*. Een compiler zet een programma wat een programmeur geschreven heeft om in machine-instructies die door de computer uitgevoerd kunnen worden. Hoe geavanceerder de programmeertaal, hoe lastiger het is om een compiler te maken.

Met attributengrammatica's kunnen compilers op een aantrekkelijke manier gemaakt worden. Programmeertalen met ingewikkelde typesystemen zijn echter lastig met attributengrammatica's te schrijven. In dit proefschrift beschouwen we uitbreidingen om attributengrammatica's ook voor het ontwikkelen van compilers voor ingewikkelde programmeertalen in te kunnen zetten.

Programmeertalen en compilers. Een computerprogramma verwerkt gegevens die zich in het geheugen van de computer bevinden. In een programmeertaal wordt deze verwerking beschreven. Voor dergelijke beschrijvingen stelt een programmeertaal elementaire verwerkingstaken ter beschikking. Tijdens de uitvoering van het programma krijgt zo'n taak gegevens uit het geheugen van de computer als invoer en laat het resultaten in het geheugen van de computer achter. Bekeken vanaf een hoog niveau kunnen we zeggen dat programmeren het samenstellen van verwerkingstaken is door de uitvoer en invoer van verwerkingstaken aan elkaar te knopen.

Een programmeertaal biedt abstractiemechanismen aan om dergelijke samenstellingen te beschrijven. Programmeren is het opstellen van zo'n beschrijving: de *broncode*. Een *compiler* vertaalt broncode naar instructies die door een computer uitgevoerd kunnen worden: de *machinecode*. Een compiler handelt details af zoals hoe de invoer en uitvoer van taken in het geheugen van de computer gerepresenteerd zijn. In de broncode is het niet nodig om dergelijke details te specificeren, wat het gemak om verwerkingstaken samen te stellen ten goede komt.

De mate waarin een programmeur een verwerkingstaak uit deeltaken kan samenstellen is van grote invloed op de kwaliteit van een programma, en de tijd die het kost om het programma te ontwikkelen en te onderhouden. Wanneer de broncode overzichtelijk is, worden er minder fouten gemaakt. Bovendien hoeft de broncode van een deeltaak maar een keer geschreven te worden. De mate waarin een programmeertaal abstractie van details toestaat speelt hierbij een belangrijke rol.

Programmeertalen bieden vaak voor specifieke domeinen speciale abstractiemechanismen aan. De taal SQL voor het raadplegen van databases is hier een goed voorbeeld van. In SQL beschrijft men het combineren van informatie uit tabellen, terwijl van de representatie van de gegevens in de tabellen en van de volgorde van het combineren geabstraheerd wordt. Idealiter richt de programmeur zich op het totaalplaatje, terwijl de compiler voor een correcte invulling van de details zorgt, eventueel aan de hand van wat expliciete hints die door de programmeur gegeven worden. Met behulp van een relatie tussen broncode en machine-instructies kan dit gedrag gespecificeerd worden.

Een programmeertaal stelt bovendien eisen aan de broncode. Bijvoorbeeld, in een samenstelling van taken dient iedere taak een correct gestructureerde invoer te hebben. Een *type* is een beschrijving van de structuur van een waarde. Ofwel, de invoer dient het juiste type te hebben. De compiler controleert als onderdeel van het vertaalproces of de broncode inderdaad aan deze eisen voldoet, en vormt dus een implementatie van het typesysteem. Een *statische semantiek* in de vorm van een *type system* specificeert deze eisen met een relatie tussen broncode en typen. Het afdwingen van deze eisen voorkomt dat bepaalde (triviale) fouten tijdens de uitvoering van het programma op kunnen treden.

Als onderdeel van het vertaalproces controleert de compiler of de broncode aan de eisen voldoet door een *bewijs* af te leiden dat de broncode relateert aan een type. De machine-instructies worden verkregen door een bewijs af te leiden dat een relatie legt met machine-instructies. Als dit niet lukt, dan bevat het programma een *statische fout* en is het programma ongeldig. Relaties in een semantiek worden doorgaans met afleidingsregels gedefinieerd. Het afleiden van zo'n bewijs wordt *inferentie* of *afleiden* genoemd. De bewijzen hebben een boomstructuur waarin de toepassing van afleidingsregels zichtbaar is.

Vrijheid in het bepalen van het bewijs geeft de compiler de mogelijkheid om details in te vullen. Echter, er bestaan harde theoretische grenzen aan wat voor bewijzen er automatisch afgeleid kunnen worden. Door de taal ingewikkelder te maken, kan er op een hoger niveau geredeneerd worden. Dan is het mogelijk een programma duidelijker uit te drukken, zodat de broncode meer structuur heeft, en er meer aannames zijn om het bewijs mee rond te krijgen. Een direct gevolg is dat de compiler daardoor lastiger wordt om te maken.

Attributengrammatica's. Als initiële stap ontleedt een compiler de broncode aan de hand van de grammatica van de programmeertaal. Het resultaat is een boomstructuur, de *abstracte syntaxboom* (AST), wat een expliciete representatie is van de compositionele structuur van de broncode. Deze boomstructuur is geschikt voor syntax-gestuurde vertaling. In dit geval heeft een semantiek een afleidingsregel voor ieder stukje syntax. De structuur van een bewijs komt dan vrijwel overeen met de AST.

Een compiler is ook een computerprogramma, en worden in een programmeertaal geschreven. Attributengrammatica's (AG's) zijn een domein-specifieke programmeertaal voor het uitdrukken van eigenschappen van ASTs, en daarmee dus ook het afleiden van bewijzen voor relaties van een syntax-gestuurde semantiek. Een AG relateert attributen met elke knoop in de AST, en specificeert functies die waarden van attributen berekenen aan de hand van waarden van andere attributen van een knoop en kinderen van de knoop. De attributen stellen eigenschappen van de broncode voor, en zijn aspecten of *getuigen* van het bewijs, zoals typen, lijsten van instructies en foutmeldingen.

De voordelen van AG's ten opzichte van algemene programmeertalen zijn dat niet beschreven wordt *hoe* de AST afgelopen wordt. Daardoor kunnen de berekeningen van attributen in afzondering beschreven worden, wat vele voordelen biedt in termen van hergebruik, overzicht en documentatie. De samenstelling van deze berekeningen wordt automatisch bepaald. Voor dit aanzienlijke voordeel is vereist dat het bewijs als attributengrammatica uit te drukken is, wat het geval is wanneer de semantiek *syntax-gestuurd* is.

Inferentie. Voor programmeertalen met een complexe (statische) semantiek is de structuur van het bewijs niet gelijk aan de AST. Tenslotte, om vrijheid in de invulling van het bewijs te hebben, dienen delen van het bewijs van de broncode afleidbaar te zijn, maar niet door de structuur ervan te worden bepaald. Daarvoor bestaan een aantal gangbare algoritmen, zoals het berekenen van een dekpunt van een stelsel van randvoorwaarden, en de gedeeltelijke verkenning van een bos van kandidaat-deelbewijzen. Deze algoritmen hebben als eigenschap dat de attributen wederzijds afhankelijk zijn van *tussentoestanden* van het bewijs. Om bijvoorbeeld een kandidaat te selecteren is het nodig om eigenschappen ervan in te zien. In een attributengrammatica zijn attributen gedefinieerd in termen van het uiteindelijke bewijs, waardoor het lastig is om dergelijke algoritmen met een AG te beschrijven.

In dit proefschrift richten we ons op *geordende* attributtengrammatica's, en breiden deze uit met de mogelijkheid om tussentoestanden te inspecteren en te manipuleren. In een geordende AG kan de berekening van de attributen als een eindige sequentie van toestandsveranderingen beschreven worden. Deze beschrijving maakt het mogelijk om over deelbewijzen in een gegeven toestand te redeneren, berekeningsstrategieën te specificeren, en attributen die in deze toestand beschikbaar zijn te inspecteren. Daarvoor schrijven we geen AGs voor de abstracte syntax van de taal, maar AGs voor de abstracte syntax van de bewijsregels van de semantiek.

Uitbreidingen. In hoofdstuk 3 introduceren we notatie om een sequentie van *visits* voor een nonterminal te specificeren. Een visit is een eenheid van evaluatie voor een knoop in de (bewijs)boom. Ieder attribuut is gerelateerd aan een visit. De attributen van een vorige visit zijn beschikbaar in een opvolgende visit. Bovendien kunnen berekeningen voor attributen aan specifieke visits toegekend worden om af te dwingen dat berekeningen in een vaste volgorde plaatsvinden. Deze uitbreiding maakt het mogelijk om monadische operaties met AGs te combineren.

In hoofdstuk 5 laten we zien hoe we berekeningsstrategiën aan visits koppelen. Door het voorwaardelijk herhalen van een visit aan een knoop kan een dekpunt berekend worden. Met *clauses* kunnen voorwaardelijke berekeningen van attributen en kinderen van de knoop gespecificeerd worden, zodat het mogelijk is om deelbewijzen te verkennen. Ook kunnen knopen ontkoppeld worden en op een andere locatie in de boom weer aangekoppeld worden, waarbij we statische garanties geven over de toestand van dergelijke verplaatsbare knopen. Zo kunnen berekeningen die afhangen van bewijzen die nog niet voltooid zijn uitgesteld worden tot deze bewijzen beschikbaar komen. Hiermee kunnen we *constraints* representeren.

In hoofdstuk 7 gaan we een stapje verder dan visits. Tussenresultaten die tijdens de uitvoering van een visit beschikbaar komen kunnen met technieken uit dit hoofdstuk stapsgewijs geïnspecteerd worden. Door om de beurt kandidaat-knopen een stap te laten zetten, kunnen de bewijzen gelijktijdig verkend worden, zonder de bewijzen van te voren al helemaal op te bouwen.

In hoodstuk 9 presenteren we AGs met *afhankelijke typen*. Dit zijn AGs waarin het type van een attribuut mag verwijzen naar de waarden van andere attributen. Deze uitbreiding maakt het mogelijk om invarianten op attributen te specificeren en bewijzen ervoor uit te drukken. Voor deze uitbreiding maken we gretig gebruik van de mogelijkheden die door voorgaande hoofdstukken besproken zijn.

De uitbreiding vormen een conservatieve uitbreiding van AGs. De mate van abstractie, zoals deze door AGs aangeboden wordt, blijft behouden. De uitbreidingen maken het mogelijk om eigenschappen van de berekeningsvolgorde te specificeren en te inspecteren, zonder daarbij het automatisch ordenen van attribuutberekeningen te breken. Met de uitbreidingen heeft de programmeur een stel krachtige bouwstenen in handen om compilers mee te implementeren.

Curriculum Vitae

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